Acceleration of Static Nash Power Control Algorithm using Newton Iterations

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Abstract— In wireless communication systems, each user's signal contributes to the interference seen by the other users. Given limited available battery power, this creates a need for effective and efficient power control strategies. These strategies may be designed to achieve quality of service (QoS) or system capacity objectives, or both. We show how the power control problem is naturally suited to formulation as a noncooperative game in which users choose to trade off between signal-to-interference ratio (SIR) error and power usage. Koskie (2003) studied the static Nash game formulation of this problem. The solution obtained led to a system of nonlinear algebraic equations. In this paper we present novel distributed power control strategies based on the use of Newton iterations, as well as variants having third-order rather than quadratic convergence, to solve the corresponding algebraic equations. The efficiency of the new algorithms is demonstrated in simulation on a realistic CDMA cell model.

I. Introduction

Closed loop power control is used in wireless communication networks to compensate for fast fading and time-varying channel characteristics, as well as to reduce mobile battery power consumption.

One of the most common approaches to closed-loop power control in wireless communication networks is SIR balancing, also called power balancing. The SIR balancing solution was originally derived for satellite communications by Aein [1] and Meyerhoff [2], and adapted for wireless communications by Nettleton [3] and Zander [4, 5]. SIR balancing algorithms are simple and most can be implemented distributively, but have the disadvantage that convergence can be slow and is guaranteed only if every mobile's target SIR is feasible.

An alternative framework for developing power control algorithms is based on game theory or economic formulations requiring the specification of a utility or cost function[6, 7, 8]. In Koskie [9], a power control algorithm using fixed point iterations was proposed for solving the system of nonlinear equations specifying the Nash equilibrium solution to the problem.

Convergence of algorithms based on fixed point iterations can be accelerated by the use of various methods, one of the simplest being the use of Newton iterations. In this paper, we present several versions of this Nash equilibrium power control algorithm based on variants of the Newton algorithm having quadratic and third-order convergence. We start by reviewing, in Sections III and IV, the development of the fixed-point Nash algorithm presented in [9]. In Section V we review the development of an accelerated algorithm based on Newton iterations with quadratic convergence presented in [12]. We then present, in Section VI, the derivation of three algorithms based on third-order variants of the Newton method. Finally, in Section VII we present simulation results of tests of these algorithms.

II. Interference in CDMA Systems

We consider the uplink direction for a single cell CDMA system with N users, designating the transmitted power and SIR for the ith user by p_i and γ_i, respectively. We denote the background (receiver) noise power within the user's bandwidth by $\eta_i := \sigma_i^2$. The interference experienced by the ith user will be designated $I_i(p_{-i})$ where the subscript "$-i$" indicates that the interference depends on the powers of all users except the ith. We use a "snapshot" model, assuming that link gains evolve slowly with respect to SIR evolution. In this problem formulation, the SIR of mobile i is given by

$$\gamma_i = \frac{g_{ii} p_i}{I_i(p_{-i})} = \frac{g_{ii} p_i}{\sum_{j \neq i} g_{ij} p_j + \eta_i} \tag{1}$$

with the interference defined by

$$I_i(p_{-i}) := \sum_{j \neq i} g_{ij} p_j + \eta_i, \tag{2}$$

where the elements g_{ij} of the gain matrix $\mathbf{G}$ are derived below. In the deterministic formulation of the power control problem for wireless networks, the noise power η_i is treated as constant.

The elements g_{ij} of the gain matrix $\mathbf{G}$ depend on mobile-to-base station distance, physical parameters, and code correlation coefficients. For the uplink, the SIR of the ith mobile is determined by

$$\gamma_i = \frac{h_i p_i}{\sum_{j \neq i} h_j p_j c_{ij} + \eta_i}, \tag{3}$$

where h_i is the attenuation from the ith mobile to the base station and c_{ij} is the code correlation coefficient. The attenuation is calculated from the distance r_i between the mobile and base station to be $h_i = A/r_i^\alpha$ in the absence of shadow and fast fading. A is a constant gain and the coefficient α is usually in the range of 3 to 4 for outdoor communications. The code correlation coefficient c_{ij} is computed from the signatures $\mathbf{s_i}$ and $\mathbf{s_j}$ to be $c_{ij} = (\mathbf{s_j}^T \mathbf{s_i})^2$. Putting these together we find that the gain matrix $\mathbf{G}$ for the uplink has elements

$$g_{ij} := \begin{cases} h_i & j = i \\ h_j (\mathbf{s_j}^T \mathbf{s_i})^2 & \text{otherwise.} \end{cases} \tag{4}$$

III. Nash Game Problem Formulation

In this section we formulate the SIR-based power control problem as a noncooperative game, choose an appropriate cost function, and find the corresponding Nash equilibrium [10, 11] power vector.

To the ith user, we assign the cost function $J_i(p_i, \gamma_i(\mathbf{p}))$ where the power vector is $\mathbf{p} := [p_1, p_2, \ldots, p_N]^T$. The Nash equilibrium strategies corresponding to these costs are the power vectors $\mathbf{p}^*$ having the property that no individual user can lower its cost by deviating from p_i^*. In other words, $\mathbf{p}^*$ satisfies

$$J_i(p_i^*, \gamma_i(\mathbf{p}^*)) \leq J_i(p_i, \gamma_i(p_1^*, p_2^*, \ldots, p_{i-1}^*, p_i, p_{i+1}^*, \ldots, p_N^*)), \quad \forall p_i, \quad \forall i = 1, 2, \ldots, N. \tag{5}$$

In general, a cost function should be convex and positive. Power always being positive in this application, a sensible choice for our cost function is

$$J_i(p_i, \gamma_i) = b_i p_i + c_i(\gamma_i^{tar} - \gamma_i)^2, \tag{6}$$

where b_i and c_i are constant weighting factors. We will see in the analysis below that only the ratio of the power weight b_i to the SIR-error weight c_i is important; hence, the c_i can be made equal to one by replacing b_i by $\tilde{b}_i = b_i/c_i$. For different applications, different ratios b_i/c_i may be chosen. Choosing $b_i/c_i > 1$ places more emphasis on power usage whereas $b_i/c_i < 1$ places more emphasis on SIR error.

Applying the necessary conditions for a Nash equilibrium we have

$$\begin{aligned} \frac{\partial J_i}{\partial p_i} = 0 &= b_i - 2c_i(\gamma_i^{tar} - \gamma_i)\frac{\partial \gamma_i}{\partial p_i} \\ &= b_i - 2c_i(\gamma_i^{tar} - \gamma_i)\left(\frac{g_{ii}}{\sum_{j \neq i} g_{ij} p_j + \eta_i}\right). \end{aligned} \tag{7}$$

Recalling that $I_i(p_{-i}) := \sum_{j \neq i} g_{ij} p_j + \eta_i$, and rearranging terms yields

$$\gamma_i = \gamma_i^{tar} - \frac{b_i I_i(p_{-i})}{2c_i g_{ii}}. \tag{8}$$

It follows from (8) that as $b_i \to 0$ (power expenditure ceases to be important) that $\gamma_i \to \gamma_i^{tar}$. Likewise, as $c_i \to 0$, (SIR value of no importance), γ_i may diverge widely from the large value γ_i^{tar}. In this latter case the only possible way to achieve high SIR's would be to set γ_i^{tar} as high as possible, which is not practical. Substituting for γ_i from (1) and isolating p_i, we can express the required power in terms of given and measured quantities as

$$p_i = \frac{\gamma_i^{tar}}{g_{ii}} I_i(p_{-i}) - \frac{b_i I_i^2(p_{-i})}{2c_i g_{ii}^2}. \tag{9}$$

Substituting for the interference using (2) in (8), and evaluating at the Nash equilibrium we have

$$\gamma_i^* = \begin{cases} \gamma_i^{tar} - \frac{b_i}{2c_i g_{ii}}\left(\frac{g_{ii} p_i^*}{\gamma_i^*}\right) & \text{if nonnegative} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Of course the equilibrium power corresponding to zero SIR is $p_i^* = 0$. Otherwise (10) yields an expression for the Nash equilibrium power p_i^* in terms of the cost weighting coefficients, the target SIR, and the Nash equilibrium SIR γ_i^*, namely

$$p_i^* = \frac{2c_i}{b_i}\gamma_i^*(\gamma_i^{tar} - \gamma_i^*). \tag{11}$$

As expected, the Nash (noncooperative) equilibrium has SIR γ_i^* less than γ_i^{tar}. When mobiles cooperate, as they must in the application of the power balancing algorithm, the target SIR, *if feasible*, will be attained by all mobiles.

IV. Fixed-Point Algorithm for Power Updates

We assume that the algorithm will run in real time using measurements that are potentially updated at every step of the algorithm. The fixed point iterations for solving (9) can be expressed in the form $p_i^{(k+1)} = f_i^{(k)}(p_i^{(k)})$ as

$$p_i^{(k+1)} = \frac{\gamma_i^{tar}}{g_{ii}} I_i^{(k)} - \frac{b_i}{2c_i}\left(\frac{I_i^{(k)}}{g_{ii}}\right)^2 =: f_i(p_i^{(k)}), \tag{12}$$

where $p_i^{(k)}$ is the power of the ith mobile and $I_i^{(k)}$ the measured interference experienced by the ith mobile at the kth step of the algorithm and $I_i^{(k)} = \sum_{j \neq i} g_{ij} p_j^{(k)} + \eta_i$. Of course the mobile power cannot be negative, so there is an implicit assumption that whenever this expression is negative, the assigned power will be zero.

For computational efficiency, we may rewrite the algorithm in terms of the previous power value $p_i^{(k)}$ and current SIR measurement $\gamma_i^{(k)}$ by

$$p_i^{(k+1)} = \gamma_i^{tar}\left(\frac{p_i^{(k)}}{\gamma_i^{(k)}}\right) - \frac{b_i}{2c_i}\left(\frac{p_i^{(k)}}{\gamma_i^{(k)}}\right)^2, \tag{13}$$

where we have substituted for $I_i^{(k)}$ in (12) using $\gamma_i^{(k)} = g_{ii} p_i^{(k)} / I_i^{(k)}$. Note that algorithm (13) differs from the power balancing algorithm in that, to the linear (in power) term, a quadratic (in power) term is added. As before, the power is required to be nonnegative.

Both formulations of the algorithm require only a single measurement at each step. Accordingly, if this measurement is made available to the mobile, either algorithm can be used to implement a distributed power control. One minor difference between the two formulations of the algorithm is that the formulation in terms of power (13), like the power balancing algorithm, requires nonzero initial powers. The formulation in terms of interference, however, does not require an initial nonzero power because the interference, which includes the noise power, is never zero.

V. Acceleration using Newton Iterations with Quadratic Convergence

A second order newton iteration algorithm for the Nash equilibrium power control was derived in [12]. This algorithm was obtained by replacing the quadratic term of the fixed-point iteration (12) by a linear approximation[13, 14],

$$(I_i^{(k+1)})^2 \quad \simeq \quad 2I_i^{(k+1)}I_i^{(k)} - (I_i^{(k)})^2. \tag{14}$$

Substituting into the quadratic term of (12) yielded the new power update equation

$$p_i^{(k+1)} - \alpha_i^{(k)} \sum_{j\neq i} g_{ij} p_{ij}^{(k+1)} = \beta_i^{(k)} + \alpha_i^{(k)} \eta_i, \tag{15}$$

where we have defined the variables

$$\alpha_i^{(k)} := \frac{1}{g_{ii}} \left(\gamma_i^{tar} - \frac{b_i}{c_i g_{ii}} I_i^{(k)} \right) \tag{16}$$

and

$$\beta_i^{(k)} := \frac{b_i}{2c_i g_{ii}^2} \left(I_i^{(k)} \right)^2 \tag{17}$$

to simplify the expression. It was shown that (15) defines a system of linear equations which can be expressed as

$$\mathbf{A}(\mathbf{p}^{(k)})\mathbf{p}^{(k+1)} = \mathbf{b}(\mathbf{p}^{(k)}), \tag{18}$$

where the i, j th element of the $\mathbf{A}(\mathbf{p}^{(k)})$ matrix is

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } i = j \\ -\alpha_i^{(k)} g_{ij} & \text{otherwise} \end{cases} \tag{19}$$

and the ith element of the $\mathbf{b}(\mathbf{p}^{(k)})$ vector is

$$b_i^{(k)} = \beta_i^{(k)} + \alpha_i^{(k)} \eta_i. \tag{20}$$

Newton iterations converge quadratically so long as a good enough initial guess is provided. For our purposes, a small positive value has been found to be acceptable.

VI. Acceleration using Third-Order Algorithms

It is well known that for a nonlinear algebraic equation of the form

$$f(x) = 0 \tag{21}$$

the Newton algorithm, defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots \tag{22}$$

with the initial condition x_0 being sufficiently close to the exact solution, has quadratic rate of convergence. It can be shown that the accelerated algorithm derived in [12] is equivalent to the vector form of this algorithm.

Recently, modifications to the Newton algorithm were presented and shown to have third-order accuracy[15, 16, 17]. The first, by Weerakoon and Fernando, uses an arithmetic mean approximation to the derivative in the denominator of (22) by an approximation[15]. Their algorithm is then

$$x_{k+1} = x_k - \frac{2f(x_k)}{f'(x_k) + f'(z_{k+1})}, \quad k = 0, 1, 2, \ldots \tag{23}$$

where

$$z_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots \tag{24}$$

Özban calls the method of (23,24) the arithmetic mean 3rd-order Newton method and proposes as an alternative, a midpoint 3rd-order Newton method defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'((x_k + z_{k+1})/2)}, \quad k = 0, 1, 2, \ldots \tag{25}$$

with z_{k+1} given by (24).

To derive third-order algorithms for the static Nash power control, we proceed as follows. From (9), recalling that the power vector is $\mathbf{p} := [p_1, p_2, \ldots, p_N]^T$, we obtain a vector equation $\mathbf{F}(\mathbf{p}) = 0$ where

$$F_i(\mathbf{p}) \quad = \quad p_i - \frac{\gamma_i^{tar}}{g_{ii}} I_i(\mathbf{p}) + \frac{b_i}{2g_{ii}^2} I_i^2(\mathbf{p}), \quad i = 1, 2, \ldots N. \tag{26}$$

Recalling also that

$$I_i(\mathbf{p}) := \sum_{j\neq i} g_{ij} p_j + \eta_i, \tag{27}$$

we obtain partial derivatives

$$\frac{\partial F_i}{\partial p_i} = 1 \tag{28}$$

and

$$\frac{\partial F_i}{\partial p_j} = -\frac{g_{ij}}{g_{ii}} \left(\gamma_i^{tar} - \frac{b_i}{c_i g_{ii}} I_i \right), \tag{29}$$

which define the derivative that we will denote by $\mathbf{F}'$. The arithmetic mean and midpoint third-order Newton algorithms are then given by

$$\mathbf{p}^{(k+1)} = \mathbf{p}^{(k)} - 2 \left(\mathbf{F}' \left(\mathbf{p}^{(k)} \right) + \mathbf{F}' \left(\mathbf{q}^{(k+1)} \right) \right)^{-1} \mathbf{F} \left(\mathbf{p}^{(k)} \right), \tag{30}$$

and

$$\mathbf{p}^{(k+1)} = \mathbf{p}^{(k)} - \left(\mathbf{F}' \left(\frac{1}{2} \left(\mathbf{p}^{(k)} + \mathbf{q}^{(k+1)} \right) \right) \right)^{-1} \mathbf{F} \left(\mathbf{p}^{(k)} \right), \tag{31}$$

respectively, where

$$\mathbf{q}^{(k)} = \mathbf{p}^{(k)} - \left(\mathbf{F}' \left(\mathbf{p}^{(k)} \right) \right)^{-1} \mathbf{F} \left(\mathbf{p}^{(k)} \right). \tag{32}$$

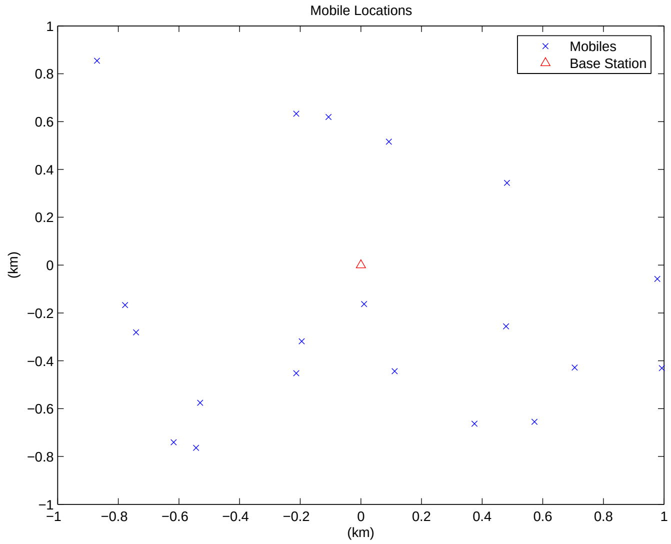

Fig. 1. A typical random distribution of 20 mobiles.

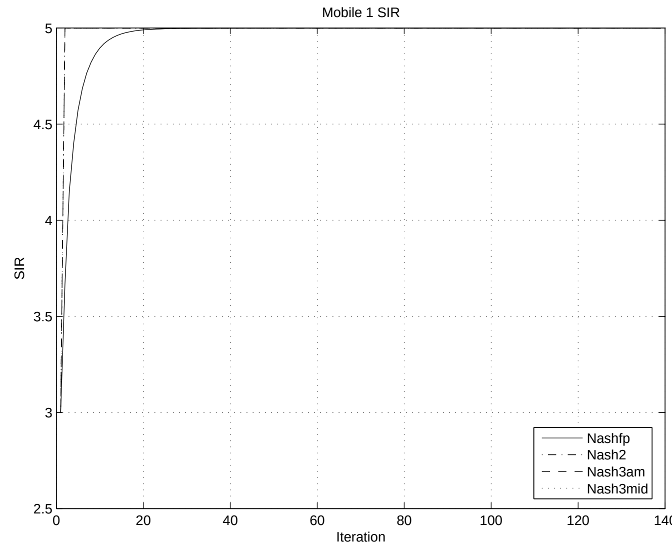

Fig. 2. SIR for a typical mobile (mobile 1) in the 20 mobile cell.

VII. Numerical Example

To demonstrate the efficacy of the algorithms developed above using Newton iterations, we tested them in a realistic simulation in Matlab. In this section we discuss the simulation parameters used, and the simulation results.

A. Simulation Parameters

We considered a 2 km square cell with base station centered at the origin and mobile locations chosen randomly from a uniform distribution[1]. A typical cell is shown in Figure 1. Power was limited to 600 mW corresponding to the legal limit in the US. Background receiver noise power within the user's bandwidth of $\sigma_i^2 = 2 \times 10^{-13}$ mW was used in the simulations. The channel gain was determined according to

$$h_i = \frac{A}{r^\alpha} \tag{33}$$

where r is the distance from the ith mobile to the base station $\alpha = 4$, and $A = 10^{-11}$, corresponding to a path loss of 110 dB at a distance of 1 km. We used random spreading sequences of length 128. We ignored fast fading and shadow fading, and interference from adjacent cells.

B. Simulation Results

We ran simulations using the fixed point algorithm (9), the accelerated version (18) having quadratic convergence, and the third-order methods (31) and (31). Results for the fully loaded cell shown in Figure 1 are shown in Figures 2 through 6. Figures 2 and 3 show the SIR and power for mobile 1. These results are typical, as indicated by Figures 4 and 5 which show the average SIR and average power over all mobiles. The differences in SIR and power values

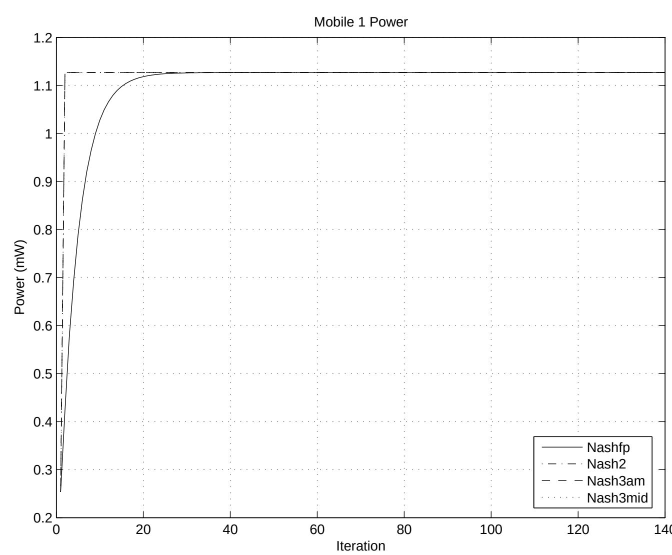

Fig. 3. Power for a typical mobile (mobile 1) in the 20-mobile cell.

[1]Realistic values for this simulation example were provided by Dr. Larry Greenstein, formerly of AT&T Bell Laboratory and currently at WINLAB, Rutgers University.

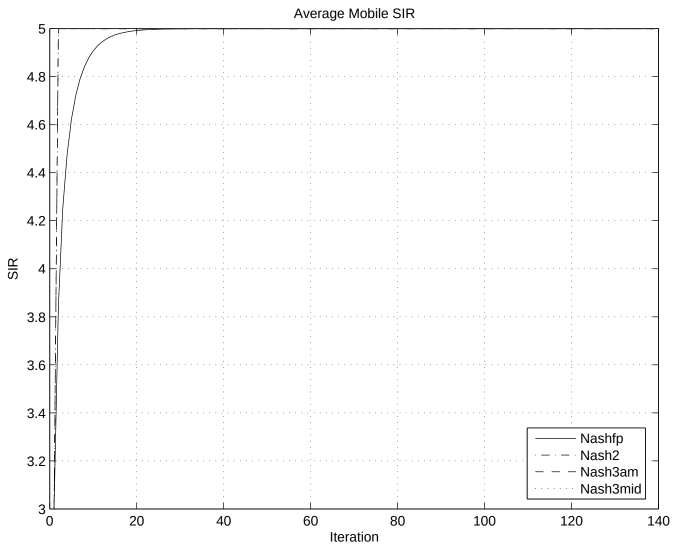

Fig. 4. Average mobile SIR in the 20-mobile cell.

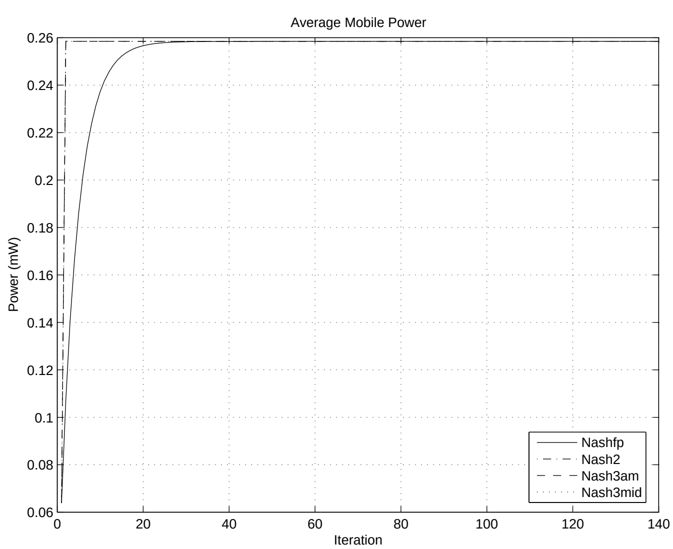

Fig. 5. Average mobile power in the 20-mobile cell.

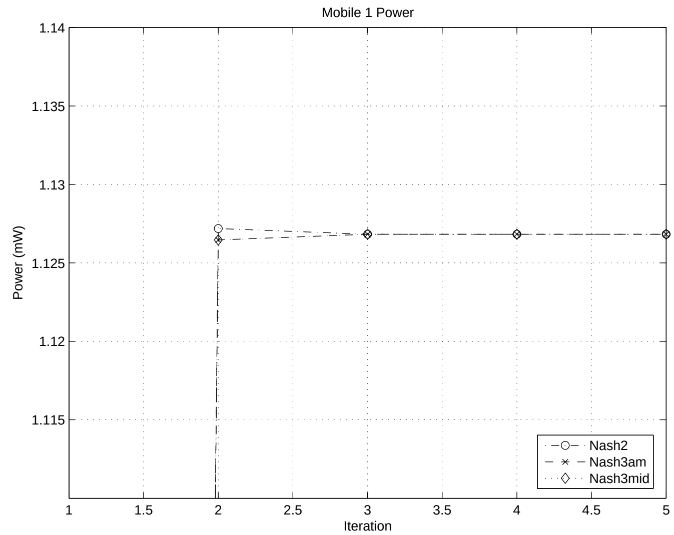

Fig. 6. Close up comparison of accelerated algorithm powers for a typical mobile (mobile 1).

resulting from the different Newton the methods are too small to be seen in these plots. Similarly, we find no difference in their convergence speeds. Specifically, if we define convergence to occur when

$$|p_i^{(k)} - p_i^{\infty}| < \epsilon, \quad \forall i \in \{1, 2, \ldots, N\} \tag{34}$$

where p_i^{∞} is the final value obtained if the algorithm is allowed to run long enough that the difference between consecutive numerical values is less than the spacing ($2.2204e-016$) of floating point numbers in Matlab (so that subtracting one from another in Matlab yields zero), we find that for $\epsilon = 1e-10$, the fixed point version of the static Nash algorithm (13) converges in 57 iterations whereas all of the accelerated versions converge in 3 iterations for this data. These simulation results were found to be typical.

To distinguish between the results of the second and third-order algorithms, we must examine the data very closely as shown in Figure 6.

When we increased the cell loading to 25 mobiles we found greater improvement in in convergence speed, as shown in Figures 7 and 8. However, for very heavily loaded cells, e.g. with $N \geq 30$, the accelerated methods sometimes failed, however for reasonable loading they were very reliable.

VIII. Conclusion

Our simulation results indicate that the use of Newton iterations to accelerate the convergence of the static Nash power control algorithm significantly decreases the number of iterations required for convergence. The advantage of the third-order algorithms over the second order algorithms appeared to consist chiefly in eliminating the slight overshoot observed in early iterations. We did not find any significant difference in the behavior of the two third-order algorithms. As noted, the accelerated algorithms are not as robust as the original fixed point algorithm for very large numbers of mobiles. Additional study would be needed to determine why the algorithms fail and to determine limits on number of allowable mobiles or other system parameters that would prevent such failures.

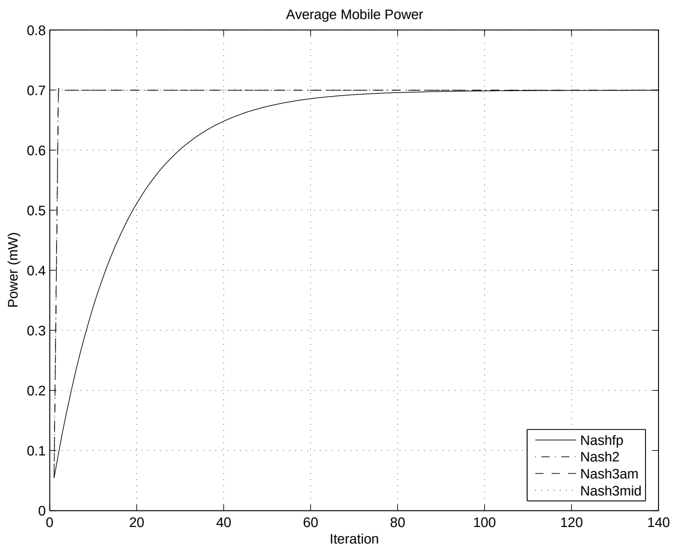

Fig. 7. Average mobile power in the 25-mobile cell.

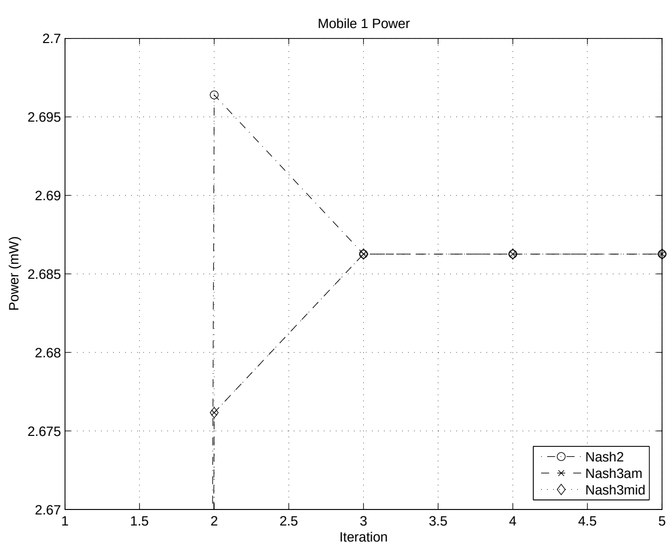

Fig. 8. Closeup of power of mobile 1 in the 25-mobile cell.

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