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CONVERGENCE PROOF FOR RECURSIVE SOLUTION OF LINEAR-QUADRATIC NASH GAMES FOR QUASI-SINGULARLY PERTURBED SYSTEMS

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Abstract. A recursive method for the solution of the Riccati equations that yield the matrix gains required for the Nash strategies for quasi-singularly perturbed systems is presented. This method involves solution of Lyapunov and reduced-order Riccati equations corresponding to reduced-order fast and slow subsystems. The algorithm is shown to converge, under specified assumptions, to the exact solution with error at the kth iteration being $O(\epsilon^k)$ where ϵ is a small, positive, singular perturbation parameter.

Keywords. Nash equilibrium, matrix algebraic Riccati equation, quasi-singularly perturbed systems, recursive algorithm, differential games

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1 Introduction

The theory of linear-quadratic Nash games for singularly perturbed systems extends results from game theory, optimization, and the theory of singular perturbations. The decomposition of singularly perturbed systems into slow and fast modes, and consequences for the design and analysis of control systems, are presented in a collection of papers edited by Kokotovic and Khalil [9]. In game theory, Starr and Ho [16] derived the optimal Nash controls for nonzero sum differential games.

Nonzero-sum closed-loop Nash games with quadratic performance cost for singularly perturbed systems combine these two areas. Conditions for the well-posedness of these problems in the infinite time case were established by Gardner and Cruz [6] and by Khalil and Kokotovic [8]. Khalil [7] defined the concept of the near-equilibrium solution and showed that the approximate strategies obtained from the truncation of the Taylor series expansion of the solution to the coupled Riccati equations constitute near-equilibrium solutions.

Potential applications of linear-quadratic Nash games for singularly perturbed systems can be found in current areas of research and applications of game theory, including economics, power systems, wireless systems and computer networks. Power systems, in particular, tend to contain subsystems that can be effectively modeled as singularly perturbed systems. A particularly interesting direction is the application of Nash games to the problem of constructing mixed H_2/H_∞ optimal controllers, presented in Limebeer *et al.* [11]. The corresponding H_2/H_∞ optimal controllers for singularly perturbed Nash Games were considered by Mukaidani *et al.* in [13].

Crucial to applications is the ability to actually compute the solutions to the coupled Riccati equations. In the case of singularly perturbed systems, recursive methods are well-suited to this task. In particular, the practicality of using recursive methods for computing the linear quadratic optimal control strategies for singularly perturbed systems has been shown by Gajic, *et al.* [3]. Recursive solution of Nash games for singularly perturbed systems was presented in Mukaidani *et al.* [12] and [13]. For quasi-singularly perturbed systems, a recursive algorithm was presented, *for the case of systems controlled exclusively through the slow modes*, in Skataric and Petrovic [15]. In this paper we generalize the algorithm to apply it to systems controlled through both fast and slow modes, provide a proof of convergence, give an estimate of the rate of convergence, and establish the required assumptions.

2 Problem Statement

2.1 Quasi-Singularly Perturbed Linear Systems

Quasi-singularly perturbed linear systems having two independent inputs can be characterized by the vector differential equation

$$\dot{x} = A(\epsilon)x + B_1(\epsilon)u_1 + B_2(\epsilon)u_2 \tag{1}$$

where $x \in \mathbb{R}^n, u_1 \in \mathbb{R}^{m_1}, u_2 \in \mathbb{R}^{m_2}, \epsilon > 0$, and

$$A(\epsilon) = \begin{bmatrix} A_1(\epsilon) & A_2(\epsilon) \\ A_3(\epsilon)/\epsilon & A_4(\epsilon)/\epsilon \end{bmatrix}, \tag{2}$$

$$B_1(\epsilon) = \begin{bmatrix} B_{11}(\epsilon) \\ B_{12}(\epsilon) \end{bmatrix}, \text{ and } \quad B_2(\epsilon) = \begin{bmatrix} B_{21}(\epsilon) \\ B_{22}(\epsilon) \end{bmatrix} \tag{3}$$

are the block matrices[1] corresponding to the decomposition of the state vector into the form $x = [x_1, x_2]^T$ with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, and $A_i(\epsilon)$ and $B_{ij}(\epsilon)$ continuously differentiable matrix functions of ϵ. This structure is used, for example, to model subsystems of a hydroelectric power plant in [14].

2.2 Nash Games

We will consider quadratic performance objectives that can be characterized by the cost functions

$$J_i(\epsilon) = \frac{1}{2}\int_0^\infty \left[x^T Q_i(\epsilon) x + u_i^T R_i(\epsilon) u_i + u_{\bar{\imath}}^T R_{\bar{\imath}i}(\epsilon) u_{\bar{\imath}}\right] dt, \quad i, \bar{\imath} \in \{1,2\},\ \bar{\imath} \neq i \tag{4}$$

with $Q_i(\epsilon) \geq 0$, $R_i(\epsilon) > 0$, and $R_{ij}(\epsilon) \geq 0$ bounded symmetric matrix functions of ϵ. Control strategies u_i^* that minimize these cost functions in the sense that

$$J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*), \qquad \forall u_1 \tag{5}$$

and

$$J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2), \qquad \forall u_2 \tag{6}$$

are called Nash strategies. The optimal linear strategies [16] are

$$u_i^* = -R_i^{-1}(\epsilon) B_i^T(\epsilon) K_i(\epsilon) x, \qquad i \in \{1,2\} \tag{7}$$

where the $K_i(\epsilon)$ are the solutions of the coupled algebraic matrix Riccati equations [2]

$$K_i A + A^T K_i + Q_i - K_i S_i K_i - K_i S_{\bar{\imath}} K_{\bar{\imath}} - K_{\bar{\imath}} S_{\bar{\imath}} K_i + K_{\bar{\imath}} Z_{\bar{\imath}} K_{\bar{\imath}} = 0 \tag{8}$$

where

$$S_i = B_i R_i^{-1} B_i^T \tag{9}$$

and

$$Z_i = B_i R_i^{-1} R_{\bar{\imath}i} R_i^{-1} B_i^T, \qquad i, \bar{\imath} \in \{1,2\},\ \bar{\imath} \neq i. \tag{10}$$

For the state weighting matrix in the quadratic costs we will use the following matrices, which are also bounded functions of the perturbation parameter ϵ,

$$Q_1(\epsilon) = \begin{bmatrix} U_1^T(\epsilon)U_1(\epsilon) & U_1^T(\epsilon)U_2(\epsilon) \\ U_2^T(\epsilon)U_1(\epsilon) & U_2^T(\epsilon)U_2(\epsilon) \end{bmatrix} \tag{11}$$

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[1] Note that this differs from the structure $B_i(\epsilon) = \begin{bmatrix} B_{i1}(\epsilon) \\ B_{i2}(\epsilon)/\epsilon \end{bmatrix}$, $i \in \{1,2\}$, of standard singularly perturbed subsystems.

[2] In (8) through (10) we have suppressed the dependence on the perturbation parameter to improve readability.

$$Q_2(\epsilon) = \begin{bmatrix} V_1^T(\epsilon)V_1(\epsilon) & V_1^T(\epsilon)V_2(\epsilon) \\ V_2^T(\epsilon)V_1(\epsilon) & V_2^T(\epsilon)V_2(\epsilon) \end{bmatrix}. \tag{12}$$

Matrices S_1 and S_2 are partitioned compatibly with the structure of the A and B matrices as [3]

$$S_i(\epsilon) \quad = \quad \begin{bmatrix} S_{i_{11}}(\epsilon) & S_{i_{12}}(\epsilon) \\ S_{i_{21}}(\epsilon) & S_{i_{22}}(\epsilon) \end{bmatrix}, \quad i \in \{1, 2\}. \tag{13}$$

The S_i matrices may have a particularly simple structure if the system is controlled exclusively through the slow modes ($B_{i2} = 0$) or exclusively through the fast modes ($B_{i1} = 0$). In the former case, only the (1,1) block of the S_i matrix is nonzero; whereas in the latter case only the (2,2) block is nonzero. We can think of such systems as being strongly controlled through the slow modes or weakly controlled through the fast modes. Examples of physical systems having such structures can be found in the literature, see *e.g.* the hydropower plant subsystem models in [14].

The Nash equations are seen to be coupled nonlinear matrix equations in the $n \times n$ matrices K_1 and K_2. These equations, as given by (8) through (10), are numerically ill-conditioned due to the structure of the A_3/ϵ and A_4/ϵ blocks of the A matrix, which makes the corresponding Jacobian matrix close to singular for small values of ϵ. This numerical ill-conditioning appears to present a formidable computational challenge. In the following section we will discuss the assumptions that allow us to achieve a significant reduction in required computation and avoid the ill-conditioning of the original problem.

2.3 Assumptions

In the Nash formulation of the problem, we will assume that there is no cross-coupling in the cost function, so $R_{\bar{\imath}i}(\epsilon) = 0$ in (4) and hence $Z_i = 0$ in (10). This corresponds to the situation where the individual cost J_i depends on the effect of both inputs on the system, but not on the value of the input $u_{\bar{\imath}}$.

As is customary in the study of singularly perturbed systems, we assume that A_4 is nonsingular. This involves no loss of generality as singularity of A_4 indicates that we have not correctly separated the slow modes from the fast modes. In addition, we will assume that the fast subsystem is stable, *i.e.* that A_4 is stable.

Two additional sets of assumptions will be required: the first to guarantee existence and uniqueness of the solutions of the system of nonlinear coupled matrix Riccati equations, and the second to ensure convergence of

[3] Note that this differs from the structure $S_i(\epsilon) = \begin{bmatrix} S_{i_{11}}(\epsilon) & S_{i_{12}}(\epsilon)/\epsilon \\ S_{i_{21}}(\epsilon)/\epsilon & S_{i_{22}}(\epsilon)/\epsilon^2 \end{bmatrix}$, $i \in \{1, 2\}$, that arises from the structure of the B_i matrices in the standard singularly perturbed case.

the algorithm. The first set will be called the stabilizability-detectability assumption (SDA). In order to state it we need some additional definitions.

To emphasize the distinction between the slow and fast modes, (see, *e.g.* [2]), we define the system matrix of the slow subsystem to be

$$A_0 := A_1^o - A_2^o A_4^{o-1} A_3^o, \tag{14}$$

where $A_i^o := A_i(0), U_i^o := U_i(0)$, and $V_i^o := V_i(0)$, and introduce the following additional variables:

$$Q_1^{(0)} := \left(U_1^o - U_2^o A_4^{o-1} A_3^o\right)^T \left(U_1^o - U_2^o A_4^{o-1} A_3^o\right) \tag{15}$$

$$Q_2^{(0)} := \left(V_1^o - V_2^o A_4^{o-1} A_3^o\right)^T \left(V_1^o - V_2^o A_4^{o-1} A_3^o\right), \tag{16}$$

which are positive semi-definite by construction.

Now we can state the slow subsystem stabilizability-detectability assumption:

The triples $\left(A_0, B_{11}, \sqrt{Q_1^{(0)}}\right)$ *and* $\left(A_0, B_{21}, \sqrt{Q_2^{(0)}}\right)$ *are stabilizable-detectable.* (SDA)

The SDA guarantees the existence and uniqueness of the nonnegative-definite stabilizing solution of the standard Riccati equation corresponding to each triple [17]. It will be used here in proving the nonsingularity of the Jacobian of the coupled algebraic Riccati equations corresponding to the coupled system. The nonsingularity of the Jacobian is required for the application of the implicit function theorem in the proof of the main result.

The second additional assumption will be a technical condition required to ensure the nonsingularity of the Jacobian. It will be presented and discussed in Section 5.1.

3 Solution

3.1 Solution Matrix Structure

If we write the Nash equation solutions $K_i(\epsilon)$ as sums of zeroth order terms and terms linear in ϵ, direct substitution into (1) through (3), shows that bounded solutions of the Nash algebraic Riccati equations have the form

$$K_1(\epsilon) = \begin{bmatrix} N_1(\epsilon) & \epsilon N_2(\epsilon) \\ \epsilon N_2^T(\epsilon) & \epsilon N_3(\epsilon) \end{bmatrix} \quad K_2(\epsilon) = \begin{bmatrix} L_1(\epsilon) & \epsilon L_2(\epsilon) \\ \epsilon L_2^T(\epsilon) & \epsilon L_3(\epsilon) \end{bmatrix}. \tag{17}$$

Note that symmetry of K_1 and K_2 requires symmetry of the block submatrices N_1, N_3, L_1, and L_3. From now on we will suppress the ϵ–dependence of the various matrix functions to improve readability.

Partitioning the Nash equations (8) compatibly with the structure of the A and B matrices (2) and (3) yields a set of six coupled algebraic matrix equations. For readability, we present the equations for the case $B_{i2} = 0$ in the text and relegate the equations for the general case to Appendix A. In the general case, the equations have the same general structure but many more terms.

$$\begin{aligned} N_1A_1 + N_2A_3 + A_1^TN_1^T + A_3^TN_2^T + U_1^TU_1 & \\ -N_1S_{1_{11}}N_1^T - N_1S_{2_{11}}L_1^T - L_1S_{2_{11}}N_1^T &= 0 \quad (18)\\ N_1A_2 + N_2A_4 + A_3^TN_3^T + U_1^TU_2 & \\ +\epsilon\big(A_1^TN_2 - N_1S_{1_{11}}N_2 - N_1S_{2_{11}}L_2 - L_1S_{2_{11}}N_2\big) &= 0 \quad (19)\\ U_2^TU_2 + N_3A_4 + A_4^TN_3^T + \epsilon\big(A_2^TN_2 + N_2^TA_2\big) & \\ +\epsilon^2\big(-N_2^TS_{1_{11}}N_2 - N_2^TS_{2_{11}}L_2 - L_2^TS_{2_{11}}N_2\big) &= 0 \quad (20)\\ L_1A_1 + L_2A_3 + A_1^TL_1^T + A_3^TL_2^T + V_1^TV_1 & \\ -L_1S_{2_{11}}L_1^T - L_1S_{1_{11}}N_1^T - N_1S_{1_{11}}L_1^T &= 0 \quad (21)\\ L_1A_2 + L_2A_4 + A_3^TL_3^T + V_1^TV_2 & \\ +\epsilon\big(A_1^TL_2 - L_1S_{2_{11}}L_2 - L_1S_{1_{11}}N_2 - N_1S_{1_{11}}L_2\big) &= 0 \quad (22)\\ V_2^TV_2 + L_3A_4 + A_4^TL_3^T + \epsilon\big(A_2^TL_2 + L_2^TA_2\big) & \\ +\epsilon^2\big(-L_2^TS_{2_{11}}L_2 - L_2^TS_{1_{11}}N_2 - N_2^TS_{1_{11}}L_2\big) &= 0 \quad (23) \end{aligned}$$

3.2 Zeroth Order Approximation

The additional terms for the general case are of order ϵ, hence the zeroth order approximation to these equations is the same for the general case as for the case of zero B_{i2}, namely:

$$Q_1^{(0)} - N_1^{(0)}S_{1_{11}}N_1^{(0)} - N_1^{(0)}S_{2_{11}}L_1^{(0)} + N_1^{(0)}A_0 - L_1^{(0)}S_{2_{11}}N_1^{(0)} + A_0^TN_1^{(0)} = 0 \quad (24)$$

$$N_2^{(0)} = -\left(N_1^{(0)}A_2 + A_3^TN_3^{(0)} + U_1^TU_2\right)A_4^{-1} \quad (25)$$

$$U_2^TU_2 + N_3^{(0)}A_4 + A_4^TN_3^{(0)} = 0 \quad (26)$$

$$Q_2^{(0)} - L_1^{(0)}S_{1_{11}}N_1^{(0)} - L_1^{(0)}S_{2_{11}}L_1^{(0)} + L_1^{(0)}A_0 - N_1^{(0)}S_{1_{11}}L_1^{(0)} + A_0^TL_1^{(0)} = 0 \quad (27)$$

$$L_2^{(0)} = -\left(L_1^{(0)}A_2 + A_3^TL_3^{(0)} + V_1^TV_2\right)A_4^{-1} \quad (28)$$

$$V_2^TV_2 + L_3^{(0)}A_4 + A_4^TL_3^{(0)} = 0. \quad (29)$$

Note that the zeroth order approximation to the partitioned Nash equations consists of a pair of coupled Nash algebraic Riccati equations in the $n_1 \times n_1$ matrix variables $N_1^{(0)}$ and $L_1^{(0)}$ corresponding to the slow subsystem;

two uncoupled Lyapunov equations in the $n_2 \times n_2$ matrix variables $N_3^{(0)}$, respectively $L_3^{(0)}$; and two uncoupled algebraic equations. We see that the zeroth order approximation consists of a Nash game for the slow subsystem only.

Note also that in the case of $B_{i1} = 0$ and hence $S_{i_{11}} = 0$ (system weakly controlled through the fast modes), the equations in $N_1^{(0)}$ and $L_1^{(0)}$ become uncoupled Lyapunov equations.

3.3 Error Equations

The power of the method to be described below arises from the structure of the error equations. We obtain the error equations by subtracting the zeroth order equations (24) through (29) from the original equations (66) through (71). The result is again a set of six coupled equations, this time in the errors between the zeroth order values and the actual values. We define the errors as follows:

$$\epsilon E_j := N_j - N_j^{(0)}, \qquad j \in \{1, 2, 3\} \tag{30}$$

$$\epsilon E_{j+3} := L_j - L_j^{(0)}, \qquad j \in \{1, 2, 3\}. \tag{31}$$

Note that symmetry of N_1, N_3, L_1, and L_3, leads to symmetry of E_1, E_3, E_4, and E_6. The errors satisfy the following set of equations:

$$D_1^{(0)T} E_1 + E_1 D_1^{(0)} + E_2 A_3 + A_3^T E_2^T - E_4 S_{2_{11}} N_1^{(0)} - N_1^{(0)} S_{2_{11}} E_4 = C_1 + \epsilon F_1(E_1, E_4) \tag{32}$$

$$E_1 A_2 + E_2 A_4 + A_3^T E_3 = C_2 + \epsilon F_2 \tag{33}$$

$$E_3 A_4 + A_4^T E_3 = C_3 + \epsilon F_3 \tag{34}$$

$$D_1^{(0)T} E_4 + E_4 D_1^{(0)} + A_3 E_5 + A_3^T E_5^T - E_1 S_{1_{11}} L_1^{(0)} - L_1^{(0)} S_{1_{11}} E_1 = C_4 + \epsilon F_4(E_1, E_4) \tag{35}$$

$$E_4 A_2 + E_5 A_4 + A_3^T E_6 = C_5 + \epsilon F_5 \tag{36}$$

$$E_6 A_4 + A_4^T E_6 = C_6 + \epsilon F_6 \tag{37}$$

where

$$D_1^{(0)} = A_1 - S_{1_{11}} N_1^{(0)} - S_{2_{11}} L_1^{(0)} \tag{38}$$

characterizes the closed-loop system and the expressions for the matrices C_j and F_j are given in Appendix B. Notice that the C_j as well as F_2, F_3, F_5, and F_6 are constant matrices in these equations. Note also that these equations, having been derived using the equations (66) through (71) where B_{i2} is potentially nonzero, hold for the general case.

The error equations (32) through (37) have an even simpler form than the zeroth order approximation. They consist of a pair of uncoupled Lyapunov equations in E_3 and E_6, a pair of algebraic equations from which E_2

and E_5 can be found in terms of the remaining error variables, and a pair of coupled Lyapunov equations in E_1 and E_4.

4 Algorithm

The following recursive algorithm, introduced in [15], exploits the structural simplification achieved by decomposing the solutions to the coupled matrix Riccati equations into a sum of zeroth order terms and error terms.

To obtain the zeroth order approximation, the linear equations (25) and (28) are solved, respectively, for $N_2^{(0)}$ in terms of $N_1^{(0)}$ and $N_3^{(0)}$, and for $L_2^{(0)}$ in terms of $L_1^{(0)}$ and $L_3^{(0)}$. These expressions are then substituted into (24) and (27). The Lyapunov equations (26) and (29) are solved for $N_3^{(0)}$ and $L_3^{(0)}$, respectively. These values are then substituted into (24) and (27) and the resulting slow subsystem coupled algebraic Riccati equations solved for $N_1^{(0)}$ and $L_1^{(0)}$. The values of $N_2^{(0)}$ and $L_2^{(0)}$ are then obtained by substituting the numerical values for $N_1^{(0)}$ and $N_3^{(0)}$, respectively, $L_1^{(0)}$ and $L_3^{(0)}$, into the respective linear equations.

The errors are obtained by the following recursive algorithm which solves a set of six linear algebraic equations. Starting with the initial condition $E_j = 0$ for all j we update the error estimates as follows:

$$
\begin{aligned}
D_1^{(0)T}E_1^+ + E_1^+D_1^{(0)} + E_2^+A_3 + A_3^TE_2^{+T} \qquad & \\
-E_4^+S_{2_{11}}N_1^{(0)} - N_1^{(0)}S_{2_{11}}E_4^+ &= C_1 + \epsilon F_1(E_1, E_4) \quad (39)\\
E_1^+A_2 + E_2^+A_4 + A_3^TE_3^+ &= C_2 + \epsilon F_2 \quad (40)\\
E_3^+A_4 + A_4^TE_3^+ &= C_3 + \epsilon F_3 \quad (41)\\
D_1^{(0)T}E_4^+ + E_4^+D_1^{(0)} + A_3E_5^+ + A_3^TE_5^{+T} \qquad & \\
-E_1^+S_{1_{11}}L_1^{(0)} - L_1^{(0)}S_{1_{11}}E_1^+ &= C_4 + \epsilon F_4(E_1, E_4) \quad (42)\\
E_4^+A_2 + E_5^+A_4 + A_3^TE_6^+ &= C_5 + \epsilon F_5 \quad (43)\\
E_6^+A_4 + A_4^TE_6^+ &= C_6 + \epsilon F_6 \quad (44)
\end{aligned}
$$

where E_j represents the current value, E_j^+ represents the next value, $N_j = N_j^{(0)} + \epsilon E_j$, and $L_j = L_j^{(0)} + \epsilon E_{j+3}$. First, the Lyapunov equations (41) and (44) are solved for E_3^+ and E_6^+. These values are then substituted into the remaining equations. The linear equations (40) and (43) are solved, respectively, for E_2^+ in terms of E_1^+ and for E_5^+ in terms of E_4^+. These expressions are substituted into the coupled Lyapunov equations (39) and (42), which are then solved for E_1^+ and E_4^+. Finally, these values are used to calculate E_2^+ and E_5^+.

5 Convergence Analysis

5.1 Nonsingularity of the Jacobian

We will now show that by recursively solving the error equations, we can obtain the solutions to the nonlinear coupled Riccati equations to arbitrary accuracy.[4] In the course of the proof we will need to apply the implicit function theorem. For this we will need to show that the Jacobian of the error equations is nonsingular, which will imply the existence of the unique and bounded solutions of the error equations.

Accordingly, we start by discussing the Jacobian. Let the vector X be the column vector resulting from expanding the blocks below to vectors and concatenating the block elements of $\tilde{X}$ where

$$\tilde{X} = \begin{bmatrix} \mathrm{vec}(E_1) \\ \mathrm{vec}(E_2) \\ \mathrm{vec}(E_3) \\ \mathrm{vec}(E_4) \\ \mathrm{vec}(E_5) \\ \mathrm{vec}(E_6) \end{bmatrix} \begin{matrix} \}\, n_1^2 \\ \}\, n_1 n_2 \\ \}\, n_2^2 \\ \}\, n_1^2 \\ \}\, n_1 n_2 \\ \}\, n_2^2 \end{matrix} \tag{45}$$

If we rearrange the error update matrix equations (32) through (37) we can write them as a single matrix equation $f(X) = 0$. Then the (i, j)th block element of the Jacobian of f evaluated at $\epsilon = 0$ is $\partial f_i / \partial E_j$, so

$$J(\epsilon)\big|_{\epsilon=0} = \begin{bmatrix} \Gamma_1 & 0 & * & \gamma_1 & 0 & 0 \\ * & \Gamma_2 & * & 0 & 0 & 0 \\ 0 & 0 & \Gamma_3 & 0 & 0 & 0 \\ \gamma_2 & 0 & 0 & \Gamma_1 & 0 & * \\ 0 & 0 & 0 & * & \Gamma_2 & * \\ 0 & 0 & 0 & 0 & 0 & \Gamma_3 \end{bmatrix} \tag{46}$$

where

$$\Gamma_1 = I_{n_1} \otimes D_0^T + D_0^T \otimes I_{n_1} \tag{47}$$

$$\Gamma_2 = I_{n_2} \otimes A_4^T \tag{48}$$

$$\Gamma_3 = I_{n_2} \otimes A_4^T + A_4^T \otimes I_{n_2} \tag{49}$$

$$\gamma_1 = I_{n_1} \otimes (-S_{2_{11}} N_1^{(0)})^T + (-S_{2_{11}} N_1^{(0)})^T \otimes I_{n_1} \tag{50}$$

$$\gamma_2 = I_{n_1} \otimes (-S_{1_{11}} L_1^{(0)})^T + (-S_{1_{11}} L_1^{(0)})^T \otimes I_{n_1} \tag{51}$$

and

$$D_0 := A_0 - S_{1_{11}} N_1^{(0)} - S_{2_{11}} L_1^{(0)} \tag{52}$$

[4]Obviously, this is contingent upon solving the equations for the zeroth order approximation to sufficient accuracy.

with I_n representing the $n \times n$ identity matrix and it will be shown that the starred entries do not appear in the determinant. (We are only interested in whether the Jacobian is nonsingular, so need not consider those block-elements that do not appear in the determinant.) Note that the Jacobian corresponds to the general case in which B_{i2} is allowed to be nonzero.

We proceed to block diagonalize the matrix in (46) by performing block row and block column exchanges followed by an application of the generalization of the Gaussian elimination (see, *e.g.* [5]) algorithm to block matrices containing a nonsingular pivot block. We find that the determinant of the Jacobian evaluated at $\epsilon = 0$ is

$$\left| J(\epsilon)|_{\epsilon=0} \right| = (-1)^{n_1^2+n_2^2} \left| \begin{bmatrix} \Gamma_1 & \gamma_1 \\ \gamma_2 & \Gamma_1 \end{bmatrix} \right| |\Gamma_3| \, |\Gamma_2| \, |\Gamma_2| \, |\Gamma_3| \tag{53}$$

with γ_1 and γ_2 defined in (50) and (51).

Stability of D_0 implies that Γ_1 is nonsingular. Because the eigenvalues of the Kronecker product of a pair of matrices are sums of eigenvalues of the matrices themselves, if a matrix is stable, *i.e.* has no eigenvalues having nonnegative real part, its Kronecker product with an identity matrix is also stable. A slightly more involved argument shows that a matrix sum of the form $I_n \otimes A + A \otimes I_n$ where A is a stable $n \times n$ matrix and I_n is the $n \times n$ identity matrix is also stable (see, *e.g.* [1]). Accordingly, stability of A_4 implies that Γ_2, and Γ_3 are nonsingular.

The Jacobian is thus nonsingular if and only if the following condition, which we will call the nonsingular Jacobian condition (NSJ), holds:

$$\textit{The } 2n_1^2 \times 2n_1^2 \textit{ matrix } \begin{bmatrix} \Gamma_1 & \gamma_1 \\ \gamma_2 & \Gamma_1 \end{bmatrix} \textit{ is nonsingular.} \tag{NSJ}$$

Note that if either $B_{1i} = 0$, the SDA implies the NSJ condition. We are now prepared to present the theorem.

5.2 Theorem Statement

Let $N_j^{(0)}$ and $L_j^{(0)}$ be solutions of the Nash algebraic Riccati equations for the slow subsystem, obtained, *e.g.* by the algorithm of Li and Gajic [10].

Let $E_j^{(k)}$ be the approximation at step k of the algorithm to the error matrix E_j. Then $N_j^{(k)}$ and $L_j^{(k)}$ are defined by

$$N_j^{(k)} := N_j^{(0)} + \epsilon E_j^{(k)} \tag{54}$$

$$L_j^{(k)} := L_j^{(0)} + \epsilon E_{j+3}^{(k)} \tag{55}$$

so at step k, the approximations to the solutions of the coupled Riccati equations are

$$K_1^{(k)} := \begin{bmatrix} N_1^{(k)} & \epsilon N_2^{(k)} \\ \epsilon N_2^{(k)T} & \epsilon N_3^{(k)} \end{bmatrix} \tag{56}$$

$$K_2^{(k)} := \begin{bmatrix} L_1^{(k)} & \epsilon L_2^{(k)} \\ \epsilon L_2^{(k)T} & \epsilon L_3^{(k)} \end{bmatrix}. \tag{57}$$

Theorem

Under the assumptions SDA, NSJ, and stability of A_4, the given algorithm converges to the exact solution of the error equations, thus providing exact solutions for $K_1(\epsilon)$ and $K_2(\epsilon)$. The rate of convergence is $O(\epsilon^k)$, *i.e.*

$$\|E_j - E_j^{(k)}\| = O(\epsilon^k), \quad \forall j \in \{1,2,3,4,5,6\}, \quad k \in \{1,2,\ldots\} \tag{58}$$

and thus

$$\|K_i - K_i^{(k)}\| = O(\epsilon^k), \quad \forall i \in \{1,2\}, \quad k \in \{1,2,\ldots\} \tag{59}$$

for any appropriate matrix norm and sufficiently small ϵ.

Proof

We will establish the result in the following sequence of claims. Claim 1 establishes that the error equations (39) through (44) can be solved. Claim 2 establishes the existence of a bounded solution of the error equation for ϵ small. Claim 3 establishes the convergence properties.

Claim 1

The error equations (39) through (44) have a unique solution at every iteration.

Proof of Claim 1

To see that the error equations (39) through (44) can be solved uniquely, we first must solve (40), respectively (43), for E_2^+, respectively E_5^+, in terms of E_1^+ and E_3^+, respectively E_4^+ and E_6^+. A_4 is stable and the right-hand side of (41), respectively (44), is symmetric, hence the unique solution E_3^+, respectively E_6^+, exists. Substituting these results into the expressions for E_2^+, respectively E_5^+, and substituting the resulting expressions into (39) and (42) we find that we have equations of the form

$$D_0^T E_1^+ + E_1^+ D_0 - E_4^+ S_{2_{11}} N_1^{(0)} - N_1^{(0)} S_{2_{11}} E_4^+ = C_1^{'} + \epsilon F_1(E_1, E_4) \tag{60}$$

and

$$D_0^T E_4^+ + E_4^+ D_0 - E_1^+ S_{1_{11}} L_1^{(0)} - L_1^{(0)} S_{1_{11}} E_1^+ = C_4^{'} + \epsilon F_4(E_1, E_4) \tag{61}$$

where the known terms containing E_3^+, respectively E_6^+, have been incorporated in the constant term $C_1^{'}$, respectively $C_4^{'}$ on the right-hand side. The resulting matrix equations in E_1^+ and E_4^+ correspond to $2n_1^2$ scalar linear equations in $2n_1^2$ unknown matrix elements. Accordingly, if we rewrite the matrix equations as discussed in Section 5.1 as $f(X) = 0$ for X the column vector obtained from $\tilde{X} = \begin{bmatrix} \mathrm{vec}(E_1^+) \\ \mathrm{vec}(E_4^+) \end{bmatrix}$, we see that the error equations have a unique solution if and only if the corresponding Jacobian is nonsingular. This Jacobian is seen to be precisely that whose nonsingularity is guaranteed by the NSJ assumption. Notice that it does not depend on the iteration, thus the coupled Lyapunov equations (39) and (42) have unique solutions at every iteration. This proves Claim 1.

Claim 2

There exists a solution of the error equations bounded in a neighborhood of $\epsilon = 0$.

Proof of Claim 2

1. We showed above that under assumption (NSJ) the Jacobian $J(\epsilon)|_{\epsilon=0}$ corresponding to (32) through (37) is nonsingular. This consisted of the following steps:
   (a) noting that the SDA implies that feedback matrices can be chosen to ensure the stability of $D_1^{(0)}$ for sufficiently small ϵ, say $\epsilon \in [0, \epsilon_D]$,
   (b) showing that properties of the Kronecker product then imply that Γ_2 and Γ_3 are stable, and
   (c) using properties of block triangular matrices to conclude that the NSJ together with Γ_2, and Γ_3 nonsingular implies the Jacobian $J(\epsilon)|_{\epsilon=0}$ is nonsingular at every iteration k.
2. Having established that the Jacobian is nonsingular at $\epsilon = 0$, we can apply the implicit function theorem to conclude the existence of an ϵ_I such that $f(E, \epsilon)$ can be solved for E as a function of ϵ for $\epsilon \in [0, \epsilon_I]$.

Accordingly, we can conclude that (32) and (35) have a unique and bounded solution for $\epsilon \in [0, \min(\epsilon_D, \epsilon_I)]$. This proves Claim 2.

Claim 3

The rate of convergence is as stated.

Proof of Claim 3

We show by induction that $\|E_j - E_j^{(k)}\| = O(\epsilon^k)$.

For the basis step, we recall that the initial values $E_j^{(0)} = 0$. We then subtract (41) from (34) to obtain

$$(E_3 - E_3^{(1)})A_4 + A_4^T(E_3 - E_3^{(1)}) = \epsilon F_3. \tag{62}$$

From the existence of bounded solutions to the error equations together with the stability of A_4 we can conclude that $\|E_3 - E_3^{(1)}\| = O(\epsilon)$. Similarly, $\|E_6 - E_6^{(1)}\| = O(\epsilon)$. Using the fact that A_4 is nonsingular we can solve (33) and (40) for E_2 and $E_2^{(1)}$, respectively and substitute the resulting expressions into (32) and (39) to obtain

$$\begin{aligned} D_0^T(E_1 - E_1^{(1)}) + (E_1 - E_1^{(1)})D_0 & \\ -(E_4 - E_4^{(1)})S_{2_{11}}N_1^{(0)} - N_1^{(0)}S_{2_{11}}(E_4 - E_4^{(1)}) &= \epsilon F_1(E_1, E_4). \end{aligned} \tag{63}$$

Likewise, we obtain a similar equation from (35) and (42). Stability of D_0 (stability of the slow subsystem in the closed loop) then allows us to conclude that $\|E_1 - E_1^{(1)}\|$ and $\|E_4 - E_4^{(1)}\|$ are $O(\epsilon)$. Then $\|E_2 - E_2^{(1)}\| = O(\epsilon)$ and $\|E_5 - E_5^{(1)}\| = O(\epsilon)$ follow similarly.

The induction step consists of showing that both $\|C_j - C_j^{(k)}\|$ and $\|F_j - F_j^{(k)}\|$ are $O(\epsilon^k)$ for all j. Since terms of the form $(N_1 S_{1_{12}} N_2^T - N_1^{(k-1)} S_{1_{12}} N_2^{(k-1)T})$ can be rewritten in the form $(N_1 S_{1_{12}}(N_2^T - N_2^{(k-1)T}) + (N_1 - N_1^{(k-1)})S_{1_{12}} N_2^{(k-1)T})$, we see that either $(N_j - N_j^{(k-1)})$ or $(L_j - L_j^{(k-1)})$, for some j, occurs as a factor of each term of the difference $C_j - C_j^{(k)}$. Thus the induction hypothesis that $\|E_j - E_j^{(k-1)}\| = O(\epsilon^{k-1})$ for all j allows us to conclude that, indeed, $\|C_j - C_j^{(k)}\| = O(\epsilon^k)$ for all j. In addition to terms of the type described above, $F_j - F_j^{(k)}$ contains terms of the form $(E_1 S_{2_{11}} E_4 - E_1^{(k-1)} S_{2_{11}} E_4^{(k-1)})$ which can be decomposed similarly. Accordingly we see that also $\|F_j - F_j^{(k)}\| = O(\epsilon^k)$ for all j. Hence we have shown by induction that $\|E_j - E_j^{(k)}\| = O(\epsilon^k)$ for all $j \in \{1, 2, 3, 4, 5, 6\}$ and $k \in \{1, 2, \ldots\}$. This completes the proof of Claim 3 and of the Theorem.

6 Suboptimal Linear Nash Strategies

The approximations $K_i^{(k)}$ defined by (56) and (57) can be used to approximate the optimal Nash strategies:

$$u_i^{(k)} = -R_i^{-1}(\epsilon)B_i^T(\epsilon)K_i^{(k)}(\epsilon)x \qquad i \in \{1, 2\}\,,\ k = 0, 1, 2, \ldots \tag{64}$$

Khalil [7] showed that in the case of analytic A, B_i, Q_i, and R_{ii}, approximating the Nash strategy by using a truncated Taylor series approximation, consisting of the first k terms of the series, yields a cost that approximates the optimal cost to $O(\epsilon^{2k})$. Since the algorithm presented above can be used to approximate the solutions to the Riccati equations to accuracy $O(\epsilon^k)$, we thus have that if we assume analyticity, this approximation also yields

$$J_i^{(k)}(u_1^{(k)}, u_2^{(k)}) = J_i(u_1^*, u_2^*) + O(\epsilon^{2k}) \quad i \in \{1,2\},\ k = 0,1,2,\ldots \tag{65}$$

If we do not assume analyticity, achieving this degree of accuracy might impose an additional limitation on the magnitude of the perturbation parameter ϵ. In either case, since the method described here can be used to compute the K_i to arbitrary accuracy, the cost can also be approximated to arbitrary accuracy.

7 Conclusions

We have shown that the Nash equilibrium controls for the quasi-singularly perturbed system can be computed by decomposing the Nash algebraic Riccati equations; solving for the zeroth order terms of the gain matrices; and recursively computing the error terms to the required accuracy. The ill-conditioning of the original matrix equations has been eliminated by decomposing the problem into subproblems corresponding to the slow and fast subsystems.

The algorithm given here is not applicable to the standard singularly perturbed system because the structure of the B_i matrices leads to entirely different zeroth order and error equations. We are currently adapting the method for this general case.

A Partitioned Nash Equations for the General Case (B_{i2} potentially nonzero)

$$\begin{aligned}
&\left(-N_2 S_{1_{22}} N_2^T - N_2 S_{2_{22}} L_2^T - L_2 S_{2_{22}} N_2^T\right)\epsilon^2 - \left(\left(N_2 S_{1_{21}} + L_2 S_{2_{21}}\right) N_1\right.\\
&\quad \left. + N_1\left(S_{1_{12}} N_2^T + S_{2_{12}} L_2^T\right) + N_2 S_{2_{21}} L_1 + L_1 S_{2_{12}} N_2^T\right)\epsilon\\
&\quad + N_1 A_1 + N_2 A_3 + A_1^T N_1 + A_3^T N_2^T + U_1^T U_1\\
&\quad - N_1 S_{1_{11}} N_1 - N_1 S_{2_{11}} L_1 - L_1 S_{2_{11}} N_1 = 0
\end{aligned} \tag{66}$$

$$\begin{aligned}
&\left(-N_2 S_{1_{21}} N_2 - N_2 S_{1_{22}} N_3 - 2 N_2 S_{2_{21}} L_2 - N_2 S_{2_{22}} L_3 - L_2 S_{2_{22}} N_3\right)\epsilon^2\\
&\quad + \left(\left(A_1^T - N_1 S_{1_{11}} - L_1 S_{2_{11}}\right) N_2 - N_1 S_{1_{12}} N_3 - N_1 S_{2_{11}} L_2 - N_1 S_{2_{12}} L_3\right.\\
&\quad \left. - L_1 S_{2_{12}} N_3\right)\epsilon + N_1 A_2 + N_2 A_4 + A_3^T N_3 + U_1^T U_2 = 0
\end{aligned} \tag{67}$$

$$\big(-N_3\,(S_{1_{21}}N_2 + S_{2_{22}}L_3 + S_{2_{21}}L_2) - N_2^T S_{1_{11}}N_2 - N_2^T S_{2_{11}}L_2 - N_3 S_{1_{22}}N_3 - N_2^T S_{2_{12}}L_3 - L_2^T S_{2_{11}}N_2 - L_3 S_{2_{21}}N_2 - \big(N_2^T S_{1_{12}} + L_2^T S_{2_{12}} + L_3 S_{2_{22}}\big)N_3\big)\epsilon^2 + \big(A_2^T N_2 + N_2^T A_2\big)\epsilon + U_2^T U_2 + N_3 A_4 + A_4^T N_3 = 0 \tag{68}$$

$$\big(-L_2 S_{2_{22}}L_2^T - L_2 S_{1_{22}}N_2^T - N_2 S_{1_{22}}L_2^T\big)\epsilon^2 - \big((L_2 S_{2_{21}} + N_2 S_{1_{21}})L_1 + L_1\big(S_{2_{12}}L_2^T + S_{1_{12}}N_2^T\big) + L_2 S_{1_{21}}N_1 + N_1 S_{1_{12}}L_2^T\big)\epsilon + L_1 A_1 + L_2 A_3 + A_1^T L_1 + A_3^T L_2^T + V_1^T V_1 - L_1 S_{2_{11}}L_1 - L_1 S_{1_{11}}N_1 - N_1 S_{1_{11}}L_1 = 0 \tag{69}$$

$$(-L_2 S_{2_{21}}L_2 - L_2 S_{2_{22}}L_3 - 2L_2 S_{1_{21}}N_2 - L_2 S_{1_{22}}N_3 - N_2 S_{1_{22}}L_3)\epsilon^2 + \big(\big(A_1^T - L_1 S_{2_{11}} - N_1 S_{1_{11}}\big)L_2 - L_1 S_{2_{12}}L_3 - L_1 S_{1_{11}}N_2 - L_1 S_{1_{12}}N_3 - N_1 S_{1_{12}}L_3\big)\epsilon + L_1 A_2 + L_2 A_4 + A_3 L_3 + V_1^T V_2 = 0 \tag{70}$$

$$\big(-L_3\,(S_{2_{21}}L_2 + S_{1_{21}}N_2 + S_{1_{22}}N_3) - L_2^T S_{2_{11}}L_2 - L_3 S_{2_{22}}L_3 - L_2^T S_{1_{11}}N_2 - L_2^T S_{1_{12}}N_3 - N_2^T S_{1_{11}}L_2 - N_3 S_{1_{21}}L_2 - \big(N_2^T S_{1_{12}} + L_2^T S_{2_{12}} + N_3 S_{1_{22}}\big)L_3\big)\epsilon^2 + \big(A_2^T L_2 + L_2^T A_2\big)\epsilon + V_2^T V_2 + L_3 A_4 + A_4^T L_3 = 0 \tag{71}$$

B Expressions for the matrices F_j and C_j

The following equations correspond to the general case with B_{i2} potentially nonzero. If B_{i2} is zero, then $S_{i_{12}}$, $S_{i_{21}}$, and $S_{i_{22}}$ are zero for $i \in \{1, 2\}$; whereas if B_{i1} is zero, then $S_{i_{11}}$, $S_{i_{12}}$, and $S_{i_{21}}$ are zero for $i \in \{1, 2\}$.

The constant matrices C_j are

$$C_1 = N_1 S_{1_{12}} N_2^T + N_2 S_{1_{21}} N_1 + N_1 S_{2_{12}} L_2^T + L_1 S_{2_{12}} N_2^T + N_2 S_{2_{21}} L_1 + L_2 S_{2_{21}} N_1 \tag{72}$$

$$C_2 = -D_1^T N_2 + N_1 S_{1_{12}} N_3 + N_1 S_{2_{11}} L_2 + N_1 S_{2_{12}} L_3 + L_1 S_{2_{12}} N_3 \tag{73}$$

$$C_3 = -N_2^T A_2 - A_2^T N_2 \tag{74}$$

$$C_4 = L_1 S_{1_{12}} N_2^T + N_1 S_{1_{12}} L_2^T + L_2 S_{1_{21}} N_1 + N_2 S_{1_{21}} L_1 + L_1 S_{2_{12}} L_2^T + L_2 S_{2_{21}} L_1 \tag{75}$$

$$C_5 = -D_1^T L_2 + L_1 S_{1_{11}} N_2 + L_1 S_{1_{12}} N_3 + N_1 S_{1_{12}} L_3 + L_1 S_{2_{12}} L_3 \tag{76}$$

$$C_6 = -L_2^T A_2 - A_2^T L_2 \tag{77}$$

where

$$D_1 := A_1 - S_{1_{11}} N_1 - S_{2_{11}} L_1. \tag{78}$$

The first order terms F_i are

$$
\begin{aligned}
F_1 &= E_1 S_{1_{11}} E_1 + E_1 S_{2_{11}} E_4 + E_4 S_{2_{11}} E_1 + N_2 S_{1_{22}} N_2^T + N_2 S_{2_{22}} L_2^T \\
&\quad + L_2 S_{2_{22}} N_2^T && (79) \\
F_2 &= N_2 S_{1_{21}} N_2 + N_2 S_{1_{22}} N_3 + N_2 S_{2_{21}} L_2 + L_2 S_{2_{21}} N_2 + N_2 S_{2_{22}} L_3 \\
&\quad + L_2 S_{2_{22}} N_3 && (80) \\
F_3 &= N_2^T S_{1_{11}} N_2 + N_2^T S_{1_{12}} N_3 + N_3 S_{1_{21}} N_2 + N_3 S_{1_{22}} N_3 \\
&\quad + N_2^T S_{2_{11}} L_2 + L_2^T S_{2_{11}} N_2 + N_2^T S_{2_{12}} L_3 + L_2^T S_{2_{12}} N_3 \\
&\quad + N_3 S_{2_{21}} L_2 + L_3 S_{2_{21}} N_2 + N_3 S_{2_{22}} L_3 + L_3 S_{2_{22}} N_3 && (81) \\
F_4 &= E_4 S_{2_{11}} E_4 + E_4 S_{1_{11}} E_1 + E_1 S_{1_{11}} E_4 + N_2 S_{1_{22}} L_2^T + L_2 S_{1_{22}} N_2^T \\
&\quad + L_2 S_{2_{22}} L_2^T && (82) \\
F_5 &= L_2 S_{1_{21}} N_2 + N_2 S_{1_{21}} L_2 + L_2 S_{1_{22}} N_3 + N_2 S_{1_{22}} L_3 + L_2 S_{2_{21}} L_2 \\
&\quad + L_2 S_{2_{22}} L_3 && (83) \\
F_6 &= L_2^T S_{1_{11}} N_2 + N_2^T S_{1_{11}} L_2 + L_2^T S_{1_{12}} N_3 + N_2^T S_{1_{12}} L_3 + L_3 S_{1_{21}} N_2 \\
&\quad + N_3 S_{1_{21}} L_2 + L_3 S_{1_{22}} N_3 + N_3 S_{1_{22}} L_3 + L_2^T S_{2_{11}} L_2 + L_2^T S_{2_{12}} L_3 \\
&\quad + L_3 S_{2_{21}} L_2 + L_3 S_{2_{22}} L_3. && (84)
\end{aligned}
$$

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