ECE 602 Lecture Notes:
Some Examples on Controllability and Observability

Example 1. Suppose that $A$ is $3 \times 3$. Is $A^4B$ linearly dependent on $\{A^3B, A^2B, AB\}$?

First, suppose that $A$ has full rank. Then $A$ is invertible so the subspace spanned by $\{A^3B, A^2B, AB\}$ is the same as that spanned by $\{A^2B, AB, B\}$. Thus every column of $A^4B$ must be in the span of $\{A^2B, AB, B\}$ and thus is linearly dependent on $\{A^2B, AB, B\}$.

Next, suppose that $A$ does not have full rank. Then $AB$ is in the range of $A$, which is not all of $\mathbb{R}^3$. We could rewrite $A^4B$ as a linear combination of columns of $\{A^2B, AB, B\}$, but if $B$ were in the null space of $A$, then we could not recover it by premultiplying it again by $A$.

Here’s an example: Suppose $A$ is

$$
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

(1)

and $B = [1 \ 1 \ 1]^T$. Then the controllability matrix

$$
C = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}
$$

has rank 2 and has two linearly independent columns

$$
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}.
$$

(2)

No matter how many more times we multiply by $A$, we’ll never get anything but zero in the third column, hence although by Cayley-Hamilton there exist real scalars $\alpha_0$, $\alpha_1$, and $\alpha_2$, $A^4B = \alpha_2A^2B + \alpha_1AB + \alpha_0B$ but $\alpha_0$ must be zero. Of course this is a very simple example, but any $n \times n$ matrix $A$ that has a nontrivial null space has a zero eigenvalue, hence its Jordan normal form can be chosen to have the zero eigenvalue in the $a_{nn}$ position, so the same argument holds.

Example 2. Consider the system

$$
A = \begin{bmatrix}
2 & 0 & -3 \\
0 & 0 & 0 \\
3/2 & 0 & 5/2
\end{bmatrix}
$$

(3)

$$
B = \begin{bmatrix}
-1 \\
0 \\
-2
\end{bmatrix}
$$

(4)

$$
C = \begin{bmatrix}
1 & 1 & -1
\end{bmatrix}.
$$

(5)
To obtain the controllability form, we first determine the controllability matrix $C$, then build an equivalence transformation from its linearly independent columns augmented to a basis.

\[
C = \begin{bmatrix}
-1 & 4 & 55/2 \\
0 & 0 & 0 \\
-2 & -13/2 & -41/4
\end{bmatrix}
\] (6)

so, for example, we can choose

\[
P^{-1} = Q = \begin{bmatrix}
-1 & 4 & 0 \\
0 & 0 & 1 \\
-2 & -13/2 & 0
\end{bmatrix}
\] (7)

Then we find $\bar{A} = PAP^{-1}$, $\bar{B} = PB$ and $\bar{C} = CP^{-1}$ or

\[
\bar{A} = \begin{bmatrix}
0 & -19/2 & 0 \\
1 & 9/2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] (8)

\[
\bar{B} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\] (9)

\[
\bar{C} = \begin{bmatrix}
1 & 21/2 & 0
\end{bmatrix}
\] (10)

which is in controllability form with $\bar{A}_c 2 \times 2$ and $\bar{A}_c 1 \times 1$.

To obtain the Kalman decomposition, we first determine whether the system is observable. The observability matrix is

\[
O = \begin{bmatrix}
1 & 1 & -1 \\
1/2 & 0 & -11/2 \\
-29/4 & 0 & -61/4
\end{bmatrix}
\] (11)

which has full rank. Hence, the system is already in Kalman form: $\bar{A}_c = \bar{A}_o$, $\bar{A}_\bar{c} = \bar{A}_\bar{o}$ and $\bar{B}_c = \bar{c}_o$, but there are no $\bar{A}_\bar{c} \bar{o}$ nor $\bar{A}_\bar{o} \bar{c}$ nor $\bar{B}_\bar{c} \bar{o}$.

**Example 3.** Consider the system

\[
A = \begin{bmatrix}
2 & 0 & -3 \\
0 & 0 & 0 \\
3/2 & 0 & 5/2
\end{bmatrix}
\] (12)

\[
B = \begin{bmatrix}
-1 \\
0 \\
-2
\end{bmatrix}
\] (13)

\[
C = \begin{bmatrix}
1 & 21/2 & 0
\end{bmatrix}
\] (14)
(Only the $C$ matrix has changed.) To obtain the Kalman decomposition, we first determine whether the system is observable. The observability matrix is

$$
\mathcal{O} = \begin{bmatrix}
1 & 0 & -1 \\
1/2 & 0 & -26/2 \\
-29/4 & 0 & -61/4 \\
\end{bmatrix}
$$

(15)

which does not have full rank. It has rank 2 so we expect to have one unobservable mode. It must either be controllable or not, so we first check the observability of the controllable subsystem. We obtain

$$
\mathcal{O}_c = \begin{bmatrix}
1 & 21/2 \\
21/2 & 149/4 \\
\end{bmatrix}
$$

(16)

which has rank 2 so the uncontrollable mode must also be unobservable. This is consistent with the fact that the third element of $\bar{C}$ is zero. Again the system in controllability form is already in Kalman form but this time, $\bar{A}_c = \bar{A}_{co}$, $\bar{A}_{\bar{c}} = \bar{A}_{\bar{c}o}$ and $\bar{B}_c = \bar{B}_{\bar{c}o}$. This time there is no $\bar{A}_{\bar{c}o}$ nor $\bar{A}_{\bar{c}o}$ nor $\bar{B}_{\bar{c}o}$. 