ECE 602 Lecture Notes: Existence and Uniqueness

We study linear continuous-time systems because they have nice properties. We know that for a linear system, the solution to the initial value problem (IVP)

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) && (1) \\ x(t_0) &= x_0 && (2) \end{aligned}$$

is the sum of the zero-input response (ZIR) and the zero-state response (ZSR).

Thus far we have simply assumed that these solutions exist. Our assumption is, in fact, true; but what would it mean for a solution not to exist? To investigate this we examine some continuous-time nonlinear systems of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), t) \\ x(t_0) &= x_0. \end{aligned} \tag{3}$$

Example 1: Solutions that exist on limited intervals

Consider

$$\begin{aligned} \dot{x}(t) &= -\frac{1}{2x(t)} \\ x(t_0) &= x_0. \end{aligned} \tag{4}$$

To solve this nonlinear ordinary differential equation (ODE) we apply the technique of separation of variables. Noting that our $f(x(t), t)$ is not defined for $x(t) = 0$, we surmise that our solution is constrained to be either positive or negative for all $t \geq t_0$. Obtaining

$$2xdx = dt, \tag{5}$$

and integrating from x_0 at t_0 to $x(t)$ at t, we have

$$\int_{x_0}^{x(t)} 2\xi d\xi = -\int_{t_0}^{t} d\tau \tag{6}$$

and thus

$$x^2(t) - x_0^2 = -t + t_0. \tag{7}$$

Now we run into a second constraint. If $x(t)$ is to be a real-valued function, then we need the constraint

$$x_0^2 - t + t_0 \geq 0. \tag{8}$$

Thus we see that if we require $t \geq t_0$ we have solutions

$$x(t) = \begin{cases} +\sqrt{x_0^2 - (t - t_0)} & x_0 > 0 \text{ and } t_0 \leq t \leq x_0^2 + t_0 \\ -\sqrt{x_0^2 - (t - t_0)} & x_0 < 0 \text{ and } t_0 \leq t \leq x_0^2 + t_0. \end{cases} \tag{9}$$

Note that no solution exists for $t > x_0^2 + t_0$.
□

Now let's examine an example in which the solution is not unique.

Example 2: A Non-unique Solution

Consider the IVP given by

$$\begin{aligned} \dot{x}(t) &= \frac{3x^{1/3}}{2} \\ x(t_0) &= x_0. \end{aligned} \tag{10}$$

Applying the separation of variables method, we obtain

$$\int_{x_0}^{x(t)} \xi^{-1/3} d\xi = \int_{t_0}^{t} \frac{3}{2} d\tau \tag{11}$$

then

$$\frac{3}{2}\left[x^{2/3}(t) - x_0^{2/3}\right] = \frac{3}{2}(t - t_0) \tag{12}$$

so, if we appropriately restrict the time interval of interest, we have a solution

$$x(t) = \left(t - t_0 + x_0^{2/3}\right)^{3/2}. \tag{13}$$

Since we are looking for a real-valued solution, the non-integer power on the right-hand side of the equation must be non-negative. Similarly, x_0 must be non-negative. Thus our solution is

$$x(t) = \left(t - t_0 + x_0^{2/3}\right)^{3/2} \qquad t \geq t_0 - x_0^{2/3} \tag{14}$$

and a solution exists iff $x_0 \geq 0$. Now let's be concrete so we can easily plot the solutions. Let $t_0 = 0$ and $x_0 = 0$. Clearly $x(t) = t^{3/2}$ is a solution to the IVP (10). However, we can also shift the solution in time. Consider a positive scalar α. For each such α,

$$x(t) = \begin{cases} 0 & 0 \leq t < \alpha \\ (t - \alpha)^{3/2} & t \geq \alpha \end{cases} \tag{15}$$

is also a solution to (10) for the initial condition $t_0 = 0$ and $x_0 = 0$. Thus for this initial condition, the IVP has uncountably many solutions.

Was there something special about the zero initial condition? The key was the condition that $x_0 = 0$ so that $\dot{x}(t_0) = 0$. The value of t_0 is arbitrary.

The general solution would be

$$x(t) = \begin{cases} 0 & t_0 \leq t < \alpha \\ (t - t_0 - \alpha)^{3/2} & t \geq \alpha \end{cases}. \tag{16}$$

□

Looking back at Example 1, we should be a little nervous about the endpoint of our time interval, because as $x(t) \rightarrow 0$, its derivative increased without bound. That's fine on paper, but would be a problem in a physical system. Does that mean that the solution is wrong? Not really. What's wrong is that the simple differential equation that we considered would cease to be an accurate model of the physical system as $x(t)$ went to zero. Doesn't look too dangerous, though, when our solution approaches zero. We could easily imagine that the rate at which our physical system state would approach zero would be limited.

Next let's examine an example that looks a little more ominous.

Example 3: Solution that Blows Up

Consider the IVP given by

$$\begin{aligned} \dot{x}(t) &= x^2 \\ x(t_0) &= x_0. \end{aligned} \tag{17}$$

Exercise 3a: Solve for arbitrary t_0 and x_0 using the method of separation of variables.

Exercise 3b: Verify that your solution satisfies the given IVP.

Exercise 3c: Determine for what values of t the solution exists.

Exercise 3d: Examine the behavior of your solution for some simple initial conditions, e.g. $(t_0, x_0) \in \{(0,0),(0,1),(1,0),(1,1)\}$.

In this example we see an example of a finite escape time – a value t_{esc} such that as $t \rightarrow t_{esc}$, $|x(t)| \rightarrow \infty$. In the case of a physical system, the differential equation will cease to be an accurate model *when something breaks.*