ECE 602 Lecture Notes: Some Notes on Lecture 21

These notes are to accompany sections 6.2, 6.6, and 6.7 of the course textbook by Chen[1].

Review of Previous Lecture

Recall that four **definitions** were presented in Lecture 20:

D1 A system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, represented by the pair $(\mathbf{A}, \mathbf{B})$, is **controllable** if for any initial condition $\mathbf{x}(0) = \mathbf{x_0}$ and any final state $\mathbf{x_1}$ there exists an input that takes $\mathbf{x_0}$ to $\mathbf{x_1}$ in a finite time.

D2 A system is **uncontrollable** if it is not controllable.

D3 The **controllability matrix** of $(\mathbf{A}, \mathbf{B})$ is

$$\mathcal{C} := \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{1}$$

D4 If $\mathbf{A}$ has all eigenvalues in the open half plane, the **controllability Gramian**

$$\mathbf{W}_c = \int_0^\infty e^{\mathbf{A}^T\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau \tag{2}$$

is the unique solution of

$$\mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}^T = -\mathbf{B}\mathbf{B}^T \tag{3}$$

and is symmetric positive definite.

We proved Theorem 6.1 which gave four **conditions** equivalent to controllability. These were

C1 The $n \times n$ matrix

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau \tag{4}$$

is nonsingular for every positive t.

C2 The $n \times np$ controllability matrix has full row rank n.

C3 The $n \times (n+p)$ matrix $\begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} & \mathbf{B} \end{bmatrix}$ has full row rank for every eigenvalue λ of $\mathbf{A}$.

C4 If all eigenvalues of $\mathbf{A}$ have negative real parts then the unique solution of (3) is positive definite.

[1]Chen, C.-T., *Linear System Theory and Design*, NY, Oxford University Press, 1999.

In the process of proving these equivalences, we used a number of pairs of equivalent conditions, specifically

E1 A positive semidefinite matrix $\mathbf{P}$ is positive definite iff it is nonsingular.

E2 A positive semidefinite matrix $\mathbf{P}$ is positive definite iff there is no nonzero vector $\mathbf{v}$ such that $\mathbf{v}^T\mathbf{P}\mathbf{v} = 0$.

E3 There exists a nonzero vector $\mathbf{v}$ such that $\mathbf{v}^T\mathbf{W}_c(t_1)\mathbf{v} = 0$ iff $\mathbf{v}^T e^{\mathbf{A}(\tau - t_1)}\mathbf{B} = 0 \ \ \forall \tau \in [0, t_1]$.

E4 For nonzero $\mathbf{v}$, $\mathbf{v}^T e^{\mathbf{A}t}\mathbf{B} = 0, \ \ \forall t > 0$ iff $\mathbf{v}^T\mathbf{A}^k\mathbf{B} = 0, \ \ \forall k = 0, 1, \ldots, n-1$. (This is a consequence of the Cayley Hamilton Theorem.)

We also showed that the input

$$\mathbf{u}(t) = -\mathbf{B}^T e^{\mathbf{A}^T(t_1 - t)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right] \tag{5}$$

takes the initial state $\mathbf{x}_0$ at $t = 0$ to the state $\mathbf{x}_1$ at $t = t_1$.

We noted that if we wanted to get from $\mathbf{x}_0$ to $\mathbf{x}_1$ twice as fast, we'd simply substitute $t_1/2$ for t_1 in (5). Thus, we could just as well get there a thousand times as fast... but that this assumed that there were no limits on the magnitude of the input. In fact, in real systems, inputs are always limited and systems become nonlinear if the inputs become too large. Thus, if we are specifying a control input for a real system, we must take care that the input remains physically achievable and that the system trajectory does not go outside the region in which the system behaves linearly.

Controllability of Discrete Time LTI Systems

The **definitions** for discrete time systems are very similar to those for continuous time.

D1' A system $\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$, represented by the pair $(\mathbf{A}, \mathbf{B})$, is **controllable** if for any initial condition $\mathbf{x}[0] = \mathbf{x_0}$ and any final state $\mathbf{x_1}$ there exists an input sequence of finite length that takes $\mathbf{x_0}$ to $\mathbf{x_1}$.

D2 A system is **uncontrollable** if it is not controllable.

D3 The **controllability matrix** of $(\mathbf{A}, \mathbf{B})$ is

$$\mathcal{C} := \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{6}$$

D4' If $\mathbf{A}$ has all eigenvalues inside the unit circle, the **controllability Gramian**

$$\mathbf{W}_c = \sum_{m=0}^{\infty} \mathbf{A}^m \mathbf{B}\mathbf{B}^T \left(\mathbf{A}^T\right)^m \tag{7}$$

is the unique solution of the discrete time Lyapunov equation

$$\mathbf{W}_c - \mathbf{A}\mathbf{W}_c\mathbf{A}^T = -\mathbf{B}\mathbf{B}^T \tag{8}$$

and is symmetric positive definite.

Equivalent **conditions** for controllability in discrete time are

C1' The $n \times n$ matrix

$$\mathbf{W}_c[n-1] = \sum_{m=0}^{n-1} \mathbf{A}^m \mathbf{B}\mathbf{B}^T \left(\mathbf{A}^T\right)^m \tag{9}$$

is nonsingular.

C2 The $n \times np$ controllability matrix $\mathbf{C}$ has full row rank n.

C3 The $n \times (n+p)$ matrix $\begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} & \mathbf{B} \end{bmatrix}$ has full row rank for every eigenvalue λ of $\mathbf{A}$.

C4' If all eigenvalues of $\mathbf{A}$ lie inside the unit circle then the unique solution of (8) is positive definite.

We can sketch the proof of the equivalences as follows:

Controllability $\Longleftrightarrow$ **C2**. The solution to the discrete time state space equation

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \tag{10}$$

is

$$\mathbf{x}[n] = \mathbf{A}^n\mathbf{x}[0] + \sum_{m=0}^{n-1} \mathbf{A}^{n-m-1}\mathbf{B}\mathbf{u}[m]. \tag{11}$$

It can be verified (exercise) that (11) can be written as

$$\mathbf{x}[n] - \mathbf{A}^n\mathbf{x}[0] = \mathcal{C} \begin{bmatrix} \mathbf{u}[n-1] \\ \mathbf{u}[n-2] \\ \vdots \\ \mathbf{u}[0] \end{bmatrix}. \tag{12}$$

C1' $\Longleftrightarrow$ **C2**. It can be verified (exercise) that (9) can be rewritten as

$$\mathbf{W}_c[n-1] = \mathcal{C} \begin{bmatrix} \mathbf{B}^T \\ \mathbf{B}^T\mathbf{A}^T \\ \vdots \\ \mathbf{B}^T\left(\mathbf{A}^T\right)^{n-1} \end{bmatrix}. \tag{13}$$

C2$\Longleftrightarrow$ **C3** was shown in the previous lecture.

C3$\Longleftrightarrow$ **C4** was shown in Chapter 5 (see Theorem 5.D6 on page 136 of the text.)

We noted that in continuous time the following are equivalent:

- Any state can be transferred to any other in finite time. (The system is said to be **controllable**.)
- Any state can be transferred to the origin. (The system is said to be **controllable to the origin**.)
- The state can be transferred from the origin to an arbitrary final state. (The system is said to be **reachable**.)

However, in discrete time **controllability to the origin** is not equivalent to **reachability** nor **controllability**. To provide a counterexample, we consider a system with a nilpotent $n \times n$ $\mathbf{A}$ and zero $\mathbf{B}$. Regardless of the initial state and input, $\mathbf{x}[n] = 0$ and the state remains at the origin forever thereafter. This system is **controllable to the origin** but it is not **reachable** nor **controllable**. Other counterexamples can be constructed easily.

Controllability of Sampled LTI Systems

Recall from Chapter 4 that if we have a piecewise constant input

$$u[k] := u(kT) = u(t) \;\; \forall t \in [kT, (k+1)T) \tag{14}$$

we can examine the sampled data system with period T,

$$\bar{\mathbf{x}}[k+1] = \bar{\mathbf{A}}\bar{\mathbf{x}}[k] + \bar{\mathbf{B}}\mathbf{u}[k] \tag{15}$$

where

$$\bar{\mathbf{A}} \;:=\; e^{\mathbf{A}T} \tag{16}$$

$$\bar{\mathbf{B}} \;:=\; \mathbf{MB} \tag{17}$$

where

$$\mathbf{M} := \int_0^T e^{\mathbf{A}t} dt. \tag{18}$$

An obvious question is whether we can find a condition in terms of $\mathbf{A}, \mathbf{B}$, and T that indicates whether the sampled data system is controllable.

Consider a pair $(\mathbf{A}, \mathbf{B})$ where, without loss of generality we assume that the $\mathbf{A}$ matrix has Jordan form consisting of a single Jordan block

$$\mathbf{A} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \tag{19}$$

Then

$$\bar{\mathbf{A}} = e^{\mathbf{A}T} = \begin{bmatrix} e^{\lambda T} & Te^{\lambda T} & T^2 e^{\lambda T}/2 \\ 0 & e^{\lambda T} & Te^{\lambda T} \\ 0 & 0 & e^{\lambda T} \end{bmatrix} \tag{20}$$

and the corresponding $\mathbf{M}$ is (exercise)

$$\mathbf{M} = \begin{bmatrix} (e^{\lambda T}-1)/\lambda & ((\lambda T-1)e^{\lambda T}+1)/\lambda^2 & ((\lambda^2 T^2 - 2\lambda T + 2)e^{\lambda T} - 2)/(2\lambda^3) \\ 0 & (e^{\lambda T}-1)/\lambda & ((\lambda T-1)e^{\lambda T}+1)/\lambda^2 \\ 0 & 0 & (e^{\lambda T}-1)/\lambda \end{bmatrix}. \tag{21}$$

It can be shown (exercise) that the Jordan normal form $\mathbf{J}$ of $\bar{\mathbf{A}}$ is

$$\mathbf{J} = \mathbf{P}\bar{\mathbf{A}}\mathbf{P}^{-1} = \begin{bmatrix} e^{\lambda T} & 1 & 0 \\ 0 & e^{\lambda T} & 1 \\ 0 & 0 & e^{\lambda T} \end{bmatrix}, \tag{22}$$

where

$$\mathbf{P} = \begin{bmatrix} T^2 e^{\lambda T^2} & 0 & 0 \\ 0 & Te^{\lambda T} & -T/2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Now with $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$, we have

$$\dot{\bar{\mathbf{x}}} = \mathbf{J}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u}, \tag{24}$$

where

$$\bar{\mathbf{B}} = \mathbf{PMB} \tag{25}$$

and the controllability matrix is

$$\begin{aligned} \bar{\mathcal{C}} &= \begin{bmatrix} \bar{\mathbf{B}} & \mathbf{J}\bar{\mathbf{B}} & \mathbf{J}^2\bar{\mathbf{B}} \end{bmatrix} && (26) \\ &= \begin{bmatrix} \mathbf{PMB} & \mathbf{P}\bar{\mathbf{A}}\mathbf{P}^{-1}\mathbf{PMB} & \mathbf{P}\bar{\mathbf{A}}^2\mathbf{P}^{-1}\mathbf{PMB} \end{bmatrix} && (27) \\ &= \mathbf{P}\begin{bmatrix} \mathbf{MB} & \bar{\mathbf{A}}\mathbf{MB} & \bar{\mathbf{A}}^2\mathbf{MB} \end{bmatrix}. && (28) \end{aligned}$$

Because $\mathbf{P}$ is invertible, $\bar{\mathcal{C}}$ has the same rank as

$$\begin{bmatrix} \mathbf{MB} & \bar{\mathbf{A}}\mathbf{MB} & \bar{\mathbf{A}}^2\mathbf{MB} \end{bmatrix}. \tag{29}$$

Because $\mathbf{A}$ in (19) is nonsingular, $\bar{\mathbf{A}}$ is nonsingular. Thus if, in addition, $\mathbf{M}$ is invertible, i.e. has full rank, $\mathbf{MB}$ will have the same column rank as $\mathbf{B}$, so $\bar{\mathcal{C}}$ will have the same row rank as

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix}. \tag{30}$$

Thus, to guarantee that $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ is controllable we would like to show that $\mathbf{M}$ is invertible. $\mathbf{M}$ being upper triangular, it will be invertible if all diagonal elements are nonzero. The diagonal elements of $\mathbf{M}$ are

$$m_{ii} := \int_0^T e^{\lambda_i \tau} d\tau = \begin{cases} (e^{\lambda_i T} - 1)/\lambda_i & \lambda_i \neq 0 \\ T & \lambda_i = 0 \end{cases} \tag{31}$$

thus the diagonal elements are nonzero exactly when there is no λ_i such that $e^{\lambda_i T} = 1$. If $\lambda_i = \alpha_i + j\beta_i$ with α_i and β_i real, then $e^{\lambda_i T} = e^{\alpha_i T} e^{j\beta_i T}$. We see that $m_{ii} = 0$ iff $\alpha_i = 0$ and $e^{j\beta_i T} = 1$. If $\beta_i T = 2k\pi$ for k an integer and $\alpha_i = 0$, then both $\pm\beta_i$ are eigenvalues so we can say that as long as we do not have both $\alpha_i = 0$ and $2\beta_i T \neq 2\pi m$ for any integer m, then the matrix $\mathbf{M}$ has no zero eigenvalues, so is nonsingular.

So far, we have considered only a single Jordan block. If the Jordan normal form of our original $\mathbf{A}$ matrix consisted of multiple blocks, we would like to show that each distinct $\lambda \in$ Spec($\mathbf{A}$) maps to a distinct $e^{\lambda T}$ in $\mathbf{J}$. Otherwise, the block structure would change. Consider two distinct eigenvalues of $\mathbf{A}$, $\lambda_i = \alpha_i + j\beta_i$ and $\lambda_k = \alpha_k + j\beta_k$. Then if $\alpha_i = \alpha_k$ and $\beta_i - \beta_k = 2m\pi/T$ for some nonzero integer m we would have two eigenvalues of $\mathbf{A}$ mapping to the same eigenvalue of $\mathbf{J}$ and the block structure of the Jordan normal form might change. (If $\alpha_i = \alpha_k$ and $m = 0$ then $\lambda_i = \lambda_k$, and the eigenvalues are not distinct.)

From this we conclude that if $(\mathbf{A}, \mathbf{B})$ is controllable and $|\beta_i - \beta_k| \neq 2\pi m/T$ for any distinct complex eigenvalue pair $\lambda_i = j\beta_i$ and $\lambda_k = j\beta_k$, then $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ is also controllable.

Miscellaneous

Is it true that the rank of $\mathcal{C}$ equals the rank of $\mathbf{A}\mathcal{C}$? Only if $\mathbf{A}$ is invertible.

Suppose that a state equation has the form

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{B_1} \\ 0 \end{bmatrix} \mathbf{u}. \tag{32}$$

Claim: Unless $(\mathbf{A}_{22}, \mathbf{A}_{21})$ is controllable, the system is not controllable.
Sketch of Proof: Consider the corresponding controllability matrix,

$$\mathcal{C} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{A}_{11}\mathbf{B}_1 & (\mathbf{A}_{11}^2 + \mathbf{A}_{12}\mathbf{A}_{21})\mathbf{B}_1 & (\mathbf{A}_{11}^3 + \mathbf{A}_{12}\mathbf{A}_{21}\mathbf{A}_{11} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{21} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{21})\mathbf{B}_1 & \cdots \\ \mathbf{0} & \mathbf{A}_{21}\mathbf{B}_1 & (\mathbf{A}_{21}\mathbf{A}_{11} + \mathbf{A}_{22}\mathbf{A}_{21})\mathbf{B}_1 & (\mathbf{A}_{21}(\mathbf{A}_{11}^2 + \mathbf{A}_{12}\mathbf{A}_{21}) + \mathbf{A}_{22}(\mathbf{A}_{21}\mathbf{A}_{11}) + \mathbf{A}_{22}^2\mathbf{A}_{21})\mathbf{B}_1 & \cdots \end{bmatrix} \tag{33}$$

The system is not controllable unless $\mathcal{C}$ has full row rank. $\mathcal{C}$ does not have full row rank unless both block rows of $\mathcal{C}$ have full row rank. Let n_2 be the number of rows in the second block row of $\mathcal{C}$. The row rank of the second block row of $\mathcal{C}$ is the same as its column rank. The column rank does not change if we subtract linear combinations of columns from each other. We note that columns of $\mathbf{A}_{21}\mathbf{A}_{11}\mathbf{B}_1$ are linear combinations of columns of $\mathbf{A}_{11}\mathbf{B}_1$, which are linear combinations of the columns of $\mathbf{B}_1$, $\mathbf{A}_{22}\mathbf{A}_{21}\mathbf{A}_{11}\mathbf{B}_1$ are linear combinations of the columns of $\mathbf{A}_{21}\mathbf{A}_{11}\mathbf{B}_1$ which are are linear combinations of columns of $\mathbf{A}_{11}\mathbf{B}_1$, which are linear combinations of columns of $\mathbf{B}_1$. The same is true for any power of $\mathbf{A}_{11}$. Thus, we can subtract $\mathbf{A}_{21}\mathbf{A}_{11}^k$, $k \geq 1$ from the kth block column of the second block row of $\mathcal{C}$. Proceeding along these lines we can subtract multiples of columns of previous columns form the kth block column to leave

$$\begin{bmatrix} \mathbf{0} & \mathbf{A}_{21} & \mathbf{A}_{22}\mathbf{A}_{21} & \mathbf{A}_{22}^2\mathbf{A}_{21} & \cdots \end{bmatrix} \mathbf{B}_1 \tag{34}$$

so if $(\mathbf{A}_{21}, \mathbf{A}_{22})$ is not controllable, then the second block row of $\mathcal{C}$ does not have full row rank, so $\mathcal{C}$ does not have full row rank.