# Some Notes on Generalized Eigenvectors

In lecture we discussed generalized eigenvalues and how to find them. At the top of page 60 the text[1] states that the representation of the matrix A with respect to the basis $v_1$, $v_2$, $v_3$, and $v_4$ is

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}. \tag{1}$$

This may be somewhat confusing since no $A$ matrix nor values for $v_1$, $v_2$, $v_3$, and $v_4$ were specified. The reason for this is as follows. $A$ here is any $4 \times 4$ matrix with the eigenstructure of $J$. It could be equal to $J$ or it could be, for example,

$$A = \begin{bmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 1 & 0 & \lambda \end{bmatrix}, \tag{2}$$

which I obtained by applying a permutation matrix to $J$. Let's start with $A = J$ and find a set of generalized eigenvectors for $A = J$. The one eigenvector corresponding to the eigenvalue $\lambda$ is $v_1 = [1 \;\; 0 \;\; 0 \;\; 0]^T$. Our next generalized eigenvector (remember that eigenvectors are considered generalized eigenvectors) should satisfy

$$(A - \lambda I)\, v_2 = v_1. \tag{3}$$

Letting $v_2 = [a \;\; b \;\; c \;\; d]^T$ we have

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

so $a$ is arbitrary, let's call it $p$, $b$ is one, and $c$ and $d$ are zero. Repeating the process we have

$$(A - \lambda I)\, v_3 = v_2. \tag{5}$$

Letting $v_3 = [a \;\; b \;\; c \;\; d]^T$ we have

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

so $a$ is arbitrary again, let's call it $q$, $b$ is $p$, $c$ is one, and $d$ is zero.

[1]Chen, C.T. *Linear System Theory and Design*, 3rd ed., Oxford University Press, 1999.

Finally, we solve

$$(A - \lambda I)\, v_4 = v_3. \tag{7}$$

Letting $v_4 = [a \;\; b \;\; c \;\; d]^T$ we have

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} q \\ p \\ 1 \\ 0 \end{bmatrix} \tag{8}$$

so $a$ is arbitrary again, let's call it $r$, $b$ is $q$, $c$ is $p$, and $d$ is one.

This is why the actual vectors $v_1$, $v_2$, $v_3$, and $v_4$ weren't specified. We can use the vectors we found as a basis and let

$$Q = \begin{bmatrix} 1 & p & q & r \\ 0 & 1 & p & q \\ 0 & 0 & 1 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

Maple tells us that

$$Q^{-1} = \begin{bmatrix} 1 & -p & p^2 - q & -p^3 + 2pq - r \\ 0 & 1 & -p & p^2 - q \\ 0 & 0 & 1 & -p \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{10}$$

and verifies that $Q^{-1} J Q = J$, regardless of the values $p$, $q$, and $r$.

Suppose we started with

$$A = \begin{bmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 1 & 0 & \lambda \end{bmatrix}, \tag{11}$$

instead. Then with help from Matlab or Maple, we verify that the characteristic equation is still $p(s) = (s - \lambda)^4$ so we have the same eigenvalues. Then

$$(A - \lambda I) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

implies that $c$ is one, $d$ is zero, and $b$ is zero so again we let $a = p$. Then

$$(A - \lambda I) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{13}$$

so $c = p$, $d = 1$, and $b = 0$, so we let $a = q$. Then

$$(A - \lambda I) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ p \\ 1 \end{bmatrix} \tag{14}$$

so $c = q$, $d = p$, $b = 1$ and we let $a = r$. Now

$$Q = \begin{bmatrix} 1 & p & q & r \\ 0 & 0 & 0 & 1 \\ 0 & 1 & p & q \\ 0 & 0 & 1 & p \end{bmatrix} \tag{15}$$

and

$$Q^{-1} = \begin{bmatrix} 1 & -r + 2qp - p^3 & -p & p^2 - q \\ 0 & p^2 - q & 1 & -p \\ 0 & -p & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{16}$$

Finally, Matlab verifies that $Q^{-1}AQ = J$.