ECE 602 Lecture Notes: Linear Spaces, Norms, and Inner Products

1. A **linear space** is defined as follows. Let F be a scalar field, *e.g.* the real numbers. Let X be a nonempty set. Given two mappings: addition mapping $X \times X$ into itself, and scalar multiplication mapping $F \times X$ into X such that the conditions below hold, X is called a linear space over F.

   (a) $x_1 + x_2 = x_2 + x_1 \ \ \forall x_1, x_2 \in X$ (commutativity of addition)

   (b) $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3 \ \ \forall x_1, x_2, x_3 \in X$ (associativity of addition)

   (c) There is a unique element 0 of X such that $0 + x = x \ \ \forall x \in X$. (existence of additive identity)

   (d) For every $x \in X$, there is a unique element $-x$ of X such that $-x + x = 0$. (existence of additive inverse)

   (e) $\alpha(\beta x) = (\alpha\beta)x \ \ \forall \alpha, \beta \in F, \ x \in X$ (associativity of scalar multiplication)

   (f) There is a unique element $1 \in F$ such that $1x = x \ \ \forall x \in X$. (existence of multiplicative identity)

   (g) There is a unique element $0 \in F$ such that $0x = 0 \ \ \forall x \in X$. (existence of zero element)

   (h) $\alpha(x_1 + x_2) = \alpha x_1 + \alpha x_2 \ \ \forall \alpha \in F, \ x_1, x_2 \in X$ (distributivity of multiplication over addition)

   (i) $(\alpha + \beta)x = \alpha x + \beta x \ \ \forall \alpha, \beta \in F, \ x \in X$ (distributivity of addition over multiplication)

2. A **norm** is a real-valued function defined on a linear space X over a field F, satisfying the conditions below. For $x \in X$, $\|x\|$ is called the norm of x.

   Nonnegativity $\|x\| \geq 0 \ \ \forall x \in X$.

   Definiteness $\|x\| = 0$ if and only if $x = 0$.

   Homogeneity $\|\alpha x\| = |\alpha| \|x\| \ \ \forall \alpha \in F, x \in X$.

   Triangle Inequality $\|x + y\| \leq \|x\| + \|y\| \ \ \forall x, y \in X$.

3. An **inner product** on a complex linear space X is a mapping from $X \times X$ to the field F of the complex numbers, satisfying the following.

   Positive Definiteness $(x, x) > 0$ for all nonzero $x \in X$.

   Additivity $(x + y, z) = (x, z) + (y, z) \ \ \forall x, y, z \in X$.

   Symmetry $(x, y) = \overline{(y, x)} \ \ \forall x, y \in X$.

   Homogeneity $(\alpha x, y) = \alpha(x, y) \ \ \forall \alpha \in F, x, y \in X$.

Note that, if F is instead the reals, then the symmetry condition becomes $(x, y) = (y, x) \ \forall x, y \in X$.

4. Comments

   An inner product generates a norm $\|x\| = (x, x)^{1/2}$.

   The Schwarz Inequality states that $|(x, y)| \leq \|x\|\|y\| \ \forall x, y \in X$.

5. Examples of Inner Products

   (a) The usual inner product on R^n is $(x, y) = \sum_{i=1}^{n} x_i y_i$.

   (b) The usual inner product on C^n is $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$.

   (c) An inner product on the space of continuous complex-valued functions on the interval $[0, 1]$ can be defined by $(x, y) = \int_0^1 x(t)\bar{y}(t)dt$.

   (d) An inner product on the space of sequences of complex numbers satisfying $\sum_{i=1}^{\infty} |x_i|^2 < \infty$ can be defined by $(x, y) = \sum_{i=1}^{\infty} x_i \bar{y}_i$.