ECE 602 Lecture Notes: Cayley-Hamilton Examples

The Cayley Hamilton Theorem states that a square $n \times n$ matrix $\mathbf{A}$ satisfies its own characteristic equation. Thus, we can express $\mathbf{A}^n$ in terms of a finite set of lower powers of $\mathbf{A}$. This fact leads to a simple way of calculating the value of a function evaluated at the matrix. This method is given in Theorem 3.5 of the textbook[1]. Here we give a couple of examples.

We will consider polynomial functions $f(\mathbf{A})$ and the exponential function $e^{\mathbf{A}t}$. We'll work with the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1}$$

which has the convenient property that

$$\mathbf{A}^k = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

The characteristic equation of $\mathbf{A}$ can be obtained easily by noting the second and third rows each have only a single nonzero entry. Using the second we have

$$\Delta(\lambda) = (s-1)^3, \tag{3}$$

so the matrix has a single eigenvalue 1 with multiplicity three.

Because the matrix is 3×3 we will use the polynomial

$$h(\lambda) := \beta_0 + \beta_1\lambda + \beta_2\lambda^2 \tag{4}$$

in our calculations. We select the notation $f^{(k)}(\lambda) := d^k f/d\lambda^k$.

Example 1 Consider the polynomial $f(\lambda) = \lambda^5 - 1$.

We obtain three linear equations in three unknowns as follows. Because the eigenvalue of $\mathbf{A}$ has multiplicity three we must use two derivatives of f and h. Our equations are

$$f(\lambda) = \lambda^5 - 1 \qquad h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 \tag{5}$$

$$f^{(1)}(\lambda) = 5\lambda^4 \qquad h^{(1)}(\lambda) = \beta_1 + 2\beta_2\lambda \tag{6}$$

$$f^{(2)}(\lambda) = 20\lambda^3 \qquad h^{(2)}(\lambda) = 2\beta_2. \tag{7}$$

[1] Chen, C.-T. *Linear System Theory and Design*, New York, Oxford University Press, 1999.

We obtain

$$f(1) = 0 = \beta_0 + \beta_1 + \beta_2 = h(1) \tag{8}$$
$$f^{(1)}(1) = 5 = \beta_1 + 2\beta_2 = h^{(1)}(1) \tag{9}$$
$$f^{(2)}(1) = 20 = 2\beta_2 = h^{(2)}(1). \tag{10}$$

Solving for the βs we find

$$\beta_2 = 10, \qquad \beta_1 = -15, \qquad \beta_0 = 5. \tag{11}$$

Thus

$$\begin{aligned} f(\mathbf{A}) &= \beta_0\mathbf{A}^0 + \beta_1\mathbf{A}^1 + \beta_2\mathbf{A}^2 && (12)\\ &= 5I - 15\mathbf{A} + 10\mathbf{A}^2 && (13)\\ &= \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} && (14)\\ &= \mathbf{A}^5 - \mathbf{I}. && (15) \end{aligned}$$

Example 2 Consider the function $f(\lambda) = e^{\lambda t}$.

We obtain three linear equations in three unknowns as follows. Because the eigenvalue of $\mathbf{A}$ has multiplicity three we must use two derivatives of f and h. Remembering that we are taking derivatives with respect to λ, our equations are

$$f(\lambda) = e^{\lambda t} \qquad h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 \tag{16}$$
$$f^{(1)}(\lambda) = te^{\lambda t} \qquad h^{(1)}(\lambda) = \beta_1 + 2\beta_2\lambda \tag{17}$$
$$f^{(2)}(\lambda) = t^2e^{\lambda t} \qquad h^{(2)}(\lambda) = 2\beta_2. \tag{18}$$

We obtain

$$f(1) = e^t = \beta_0 + \beta_1 + \beta_2 = h(1) \tag{19}$$
$$f^{(1)}(1) = te^t = \beta_1 + 2\beta_2 = h^{(1)}(1) \tag{20}$$
$$f^{(2)}(1) = t^2e^t/2 = 2\beta_2 = h^{(2)}(1). \tag{21}$$

Solving for the βs we find

$$\beta_2 = t^2e^t/2, \qquad \beta_1 = te^t - t^2e^t, \qquad \beta_0 = e^t - te^t + t^2e^t/2. \tag{22}$$

Thus

$$\begin{aligned} f(\mathbf{A}) &= \beta_0\mathbf{A}^0 + \beta_1\mathbf{A}^1 + \beta_2\mathbf{A}^2 && (23)\\ &= e^t\left(1 - t + t^2/2\right)I + e^t\left(t - t^2\right)\mathbf{A} + e^t\left(t^2/2\right)\mathbf{A}^2 && (24)\\ &= \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}. && (25) \end{aligned}$$

Example 3 Computation of $(s\mathbf{I} - \mathbf{A})^{-1}$ for

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

using equivalence on the spectrum of $\mathbf{A}$.

First let's find the spectrum of $\mathbf{A}$.

$$|sI - \mathbf{A}| = \left| \begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix} \right| = s(s+2) - (-1) = s^2 + 2s + 1 = (s+1)^2$$

so $\mathbf{A}$ has eigenvalue $\lambda = -1$ with multiplicity 2.

Next, let $f(\lambda) = (s - \lambda)^{-1}$ and, as always, $h(\lambda) = \beta_0 + \beta_1 \lambda$.

Then $f^{(1)}(\lambda) = -(-1)(s - \lambda)^{-2}$ and $h^{(1)}(\lambda) = \beta_1$.

Substituting the eigenvalue for λ yields

$$\beta_1 = (s+1)^{-2} \quad (26)$$

$$\beta_0 = (s+1)^{-1} - (-1)(s+1)^{-2} = (s+2)(s+1)^{-2} \quad (27)$$

Thus $h(\lambda) = (s+2)(s+1)^{-2} + (s+1)^{-2}\lambda$ and substituting $\mathbf{A}$ for λ yields

$$f(\mathbf{A}) = (s\mathbf{I} - \mathbf{A})^{-1} \quad (28)$$

$$= (s+2)(s+1)^{-2}\mathbf{I} + (s+1)^{-2}\mathbf{A} \quad (29)$$

$$= \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{(s+2-2)}{(s+1)^2} \end{bmatrix}. \quad (30)$$