ECE 602 Lecture Notes:
Examination of a Companion Matrix

Yesterday in lecture we examined the companion matrix

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{1}$$

The characteristic equation of A is

$$\begin{aligned} \det(sI - A) &= \begin{vmatrix} s+\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & s \end{vmatrix} \\ &= (s+\alpha_1)\begin{vmatrix} s & 0 & 0 \\ -1 & s & 0 \\ 0 & -1 & s \end{vmatrix} - (-1)\begin{vmatrix} \alpha_2 & \alpha_3 & \alpha_4 \\ -1 & s & 0 \\ 0 & -1 & s \end{vmatrix} \\ &= (s+\alpha_1)\, s^3 + \alpha_2 s^2 - (-1)\left(\alpha_3 s - (-\alpha_4)\right) \\ &= s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4 = 0. \end{aligned} \tag{2}$$

Thus, if α_4 were zero, the characteristic polynomial would be

$$s\left(s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3\right)$$

and zero would be an eigenvalue of A. If zero were an eigenvalue of A, then A would be singular, *i.e.* non-invertible.

Can the matrix be singular if α_4 is nonzero? The answer is "no", because if α_4 is nonzero, then the four rows of the matrix A are linearly independent regardless of the values of α_1, α_2, and α_3. Of course we could also have used this simpler argument to establish that if α_4 were zero, the matrix A would be singular. Specifically, if α_4 were zero, the fourth column would be the zero vector so we would have only three linearly independent rows/columns.

To show that

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix}, \tag{3}$$

we simply multiply A by A^{-1} and verify that the result is I.

What can we say about the eigenvalues and eigenvectors?

Suppose that λ is an eigenvalue of A. Then

$$\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0$$

Also,

$$\begin{aligned} 0 &= (A - \lambda I)\, v \\ &= \begin{bmatrix} -\alpha_1 - \lambda & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \\ &= \begin{bmatrix} (-\alpha_1 - \lambda)v_1 - \alpha_2 v_2 - \alpha_3 v_3 - \alpha_4 v_4 \\ v_1 - \lambda v_2 \\ v_2 - \lambda v_3 \\ v_3 - \lambda v_4 \end{bmatrix} \end{aligned} \tag{4}$$

Here we have (assuming λ known) four equations in the four unknowns v_1, v_2, v_3, and v_4. From the fourth row we have that $v_3 = \lambda v_4$. From the third, $v_2 = \lambda v_3 = \lambda^2 v_4$. From the second, $v_1 = \lambda v_2 = \lambda^2 v_3 = \lambda^3 v_4$. Then the first row is

$$\begin{aligned} 0 &= (-\alpha_1 - \lambda)v_1 - \alpha_2 v_2 - \alpha_3 v_3 - \alpha_4 v_4 \\ &= (-\alpha_1 - \lambda)\lambda^3 v_4 - \alpha_2 \lambda^2 v_4 - \alpha_3 \lambda v_4 - \alpha_4 v_4 \\ &= \left((-\alpha_1 - \lambda)\lambda^3 - \alpha_2\lambda^2 - \alpha_3\lambda - \alpha_4\right) v_4 \\ &= -\left(\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4\right) v_4. \end{aligned} \tag{5}$$

We already know that if λ is an eigenvalue of A it satisfies the characteristic equation of A so this first equation holds regardless of the value of v_4. As always, we can choose one element of the eigenvector arbitrarily so let's make $v_4 = 1$. Then the eigenvector corresponding to λ is

$$v = \begin{bmatrix} \lambda^3 & \lambda^2 & \lambda & 1 \end{bmatrix}^T.$$