Slack, Surplus, and Free Variables

To convert an equation of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{1n}x_n \leq b_i$$

to standard form we introduce a **slack** variable y_i to obtain

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{1n}x_n + y_i = b_i.$$

To convert an equation of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{1n}x_n \geq b_i$$

to standard form we introduce a **surplus** variable y_i to obtain

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{1n}x_n - y_i = b_i.$$

To deal with a variable that is not constrained to be nonnegative we can use one of two approaches. Suppose that x_i is not constrained to be nonegative. In this case x_i is called a **free variable**.

1. Let
$$x_i = u_i - v_i, \qquad \text{where} \quad u_i \geq 0 \quad \text{and} \quad v_i \geq 0.$$
Then replace x_i by $u_i - v_i$ everywhere that it occurs in the objective function and the rows of $Ax = b$. Note that this method introduces redundancy.

2. Select a row in which x_i has a nonzero coefficient and solve it for x_i. Then substitute the expression obtained for x_i everywhere it occurs in the objective function and the rows of $Ax = b$. Note that the ith equation becomes trivial, namely, $b_i = b_i$.

Example

Put the LP problem

$$\begin{array}{rl} \min & x_1 + 3x_2 + 4x_3 \\ \text{subject to} & x_1 + 2x_2 + x_3 = 5 \\ & 2x_1 + 3x_2 + x_3 = 6 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \end{array}$$

in standard form. Since x_1 is a free variable we can use either of the two free variable methods.

Method 1:

Introduce the two new nonegative variables u_1 and v_1 to obtain

$$x_1 = u_1 - v_1. \tag{1}$$

Substitute for x_1 everywhere. The objective function becomes $f(x) = c^T x$ where

$$c = \begin{bmatrix} 3 & 4 & 1 & -1 \end{bmatrix}^T \tag{2}$$

$$x = \begin{bmatrix} x_2 & x_3 & u & v \end{bmatrix}^T \tag{3}$$

Substituting into the constraints we obtain

$$u_1 - v_1 + 2x_2 + x_3 = 52(u_1 - v_1) + 3x + 2 + x + 3 = 6. \tag{4}$$

Thus we have the standard form LP

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

where

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -2 & 3 & 1 \end{bmatrix} \tag{5}$$

$$b = \begin{bmatrix} 5 & 6 \end{bmatrix}^T \tag{6}$$

$$c = \begin{bmatrix} 3 & 4 & 1 & -1 \end{bmatrix}^T \tag{7}$$

$$x = \begin{bmatrix} x_2 & x_3 & u & v \end{bmatrix}^T. \tag{8}$$

Method 2:

Solve one of the constraint equations for x_1 and substitute it into the objective function and the remaining constraint equations. We'll solve the first for x_1 to obtain

$$x_1 = 5 - 2x_2 - x_3, \tag{9}$$

substitute it into the objective function to obtain

$$5 - 2x_2 - x_3 + 3x_2 + 4x_3 = x_2 + 3x_3 + 5, \tag{10}$$

and substitute into the second to obtain from

$$2(5 - 2x_2 - x_3) + 3x_2 + x_3 = 6, \tag{11}$$

$$-x_2 - x_3 = -4. \tag{12}$$

Note that minimizing $x_2 + 3x_3 + c$ for any scalar c is equivalent to minimizing $x_2 + 3x_3$ so we can state the objective as minimizing $x_2 + 3x_3$. Note also that we are supposed to have $b \geq 0$, so we need to multiply (**??**) by -1. Thus we have the LP in standard form

$$\begin{array}{rl} \min & x_2 + 3x_3 \\ \text{subject to} & x_2 + x_3 = 4 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \end{array}$$

or in matrix form

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

where

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{13}$$

$$b = 4 \tag{14}$$

$$c = \begin{bmatrix} 1 & 3 \end{bmatrix}^T \tag{15}$$

$$x = \begin{bmatrix} x_2 & x_3 \end{bmatrix}^T. \tag{16}$$

We note that the second method gives us a more efficient statement of the LP.

Two Ways to solve Example 15.11

Problem Statement: Convert the optimization problem

$$\begin{array}{rl} \text{maximize} & x_2 - x_1 \\ \text{subject to} & 3x_1 = x_2 - 5 \\ & |x_2| \leq 2 \\ & x_1 \leq 0 \end{array}$$

to a linear program in standard form.

Solution 1 (presented in class):

1. In order to state the problem as a minimization problem we must replace $f(x)$ by $-f(x)$.
2. In order to be able to write the final constraint as $x \geq 0$ we must replace x_1 by $\tilde{x}_1 = -x_1$. Thus we have $\tilde{f}(x) = -\tilde{x}_1 - x_2$.
3. After substituting $-\tilde{x}_1$ for x_1 in the first constraint, we have $-\tilde{x}_1 - x_2 = -5$. Note that this equation is already an equality constraint so we do not need to uintroduce any new variables. However, it is customary to require $b \geq 0$ so we will multiply this equation by -1 to obtain $\tilde{x}_1 + x_2 = 5$.

4. Replace $|x_2| \leq 2$ by two inequalities $x_2 \leq 2$ and $-2 \leq x_2$. We insert a slack variable in the first inequality to obtain $x_2 + y_1 = 2$ where $y_1 \geq 0$. We insert a surplus variable in the second to obtain $x_2 - y_2 = -2$, where $y_2 \geq 0$. Again, to obtain $b \geq 0$ we multiply this second equation by -1. So far we have the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{17}$$

$$b = \begin{bmatrix} 5 & 2 & 2 \end{bmatrix} \tag{18}$$

$$c = \begin{bmatrix} -1 & -10 & 0 \end{bmatrix}^T \tag{19}$$

$$x = \begin{bmatrix} \tilde{x}_1 & x_2 & y_1 & y_2 \end{bmatrix}^T; \tag{20}$$

however, we do not have $x_2 \geq 0$. So we need one more step.

5. We are left with free variable x_2. To eliminate it we solve one of our equations for x_2 and substitute for x_2 in the others.

   From the last equation we have $x_2 = y_2 - 2$. Then the first equation becomes $\tilde{x}_1 + y_2 = 7$. The second equation becomes $y_2 + y_1 = 4$ and the third becomes trivial. The objective function becomes $f(x) = -\tilde{x}_1 - (y_2 - 2)$. However, minimizing $f(x) + \alpha$ for a scalar α is equivalent to minimizing $f(x)$ so we can write simply $f(x) = -\tilde{x}_1 - y_2$.

We now have

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{21}$$

$$b = \begin{bmatrix} 7 & 4 \end{bmatrix} \tag{22}$$

$$c = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}^T \tag{23}$$

$$x = \begin{bmatrix} \tilde{x}_1 & y_1 & y_2 \end{bmatrix}^T. \tag{24}$$

Solution 2 (details following the outline in the tex):

Note that steps 1 through 4 are identical to the first method shown above.

In step 5, instead of eliminating the free variable x_2, the authors introduce two new nonnegative variables u and v and take $x_2 = u - v$. Substituting for x_2 everywhere then yields

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

where

$$A = \begin{bmatrix} 3 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \tag{25}$$

$$b = \begin{bmatrix} 5 & 2 & 2 \end{bmatrix} \tag{26}$$

$$c = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \end{bmatrix}^T \tag{27}$$

$$x = \begin{bmatrix} \tilde{x_1} & y_1 & y_2 & u & v \end{bmatrix}^T, \tag{28}$$

where our y_1 is the textbook's x_3 and our y_2 is the textbook's x_4.