Newton and Secant Methods

The following notes are an attempt to capsulize the algorithms of sections 7.3 and 7.4 of our textbook by Chong and Zak.

Algorithms

Newton's Method (scalar case) for finding a minimizer

Newton's method starts with the first 3 terms of the Taylor series of f,

$$f(x) \sim f(x^{(k)}) + f'(x^{(k)})(x - -x^{(k)}) + \frac{1}{2}f''(x^{(k)})(x - -x^{(k)})^2. \tag{1}$$

To find the minimizer of the approximation, we solve the First Order Necessary Condition (FONC)

$$f'(x) = f'(x^{(k)}) + f''(x^{(k)})(x - x^{(k)}) \tag{2}$$

for x to obtain the Newton algorithm.

Given a function $f(x)$ and an initial guess $x^{(0)}$, if $x^{(0)}$ is close enough to the minimizer of f, the sequence given by

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})} \tag{3}$$

will converge to a minimizer of f.

Newton's Method of Tangents (scalar case) for finding a root

Given a function $g(x)$ for which we wish to solve $g(x) = 0$, if $x^{(0)}$ is close enough to a root of g, the sequence given by

$$x^{(k+1)} = x^{(k)} - \frac{g(x^{(k)})}{g'(x^{(k)})} \tag{4}$$

will converge to a root of g.

Secant Method (scalar case) for finding a minimizer

The secant method approximates $f''(x^{(k)})$ in the Newton algorithm by

$$f''(x^{(k)}) = \frac{f'(x^{(k)}) - f'(x^{(k-1)})}{x^{(k)} - x^{(k-1)}}, \tag{5}$$

to obtain a sequence of $x^{(k)}$ converging to a minimizer of f. The secant algorithm is

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})\left(x^{(k)} - x^{(k-1)}\right)}{f'(x^{(k)}) - f'(x^{(k-1)})}. \tag{6}$$

Secant Method (scalar case) for finding a root

The algorithm for finding a root of a funtion $g(x)$ is

$$x^{(k+1)} = x^{(k)} - \frac{g(x^{(k)})\left(x^{(k)} - x^{(k-1)}\right)}{g(x^{(k)}) - g(x^{(k-1)})}. \tag{7}$$

Examples

These examples correspond to problems 8, 10, 11, and 12 of the Fall 2010 midterm exam.

Problem 8 (Newton's method for finding a minimizer (vector case))

We are given

$$f(x) = 2x_1^2 + 4x_1 + x_2^2 - 2x_2 \qquad \text{with} \qquad x^{(0)}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{8}$$

and asked to find $x^{(1)})$. We will need the first and second derivatives (formulae on p. 81)

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_1} \end{bmatrix} = (\nabla f)^T = \begin{bmatrix} 4x_1 + 4 & 2x_2 - 2 \end{bmatrix}. \tag{9}$$

and

$$D^2 f = \mathbf{F}(x) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}. \tag{10}$$

$D^2 f$ is positive definite because its eigenvalues (on the diagonal) are positive. Thus

$$\begin{aligned} x^{(1)} &= x^{(0)} - (D^2 f(x^{(0)}))^{-1} \nabla f(x^{(0)}) && (11) \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 4x_1 + 4 \\ 2x_2 - 2 \end{bmatrix} \Bigg|_{x=x^{(0)}} && (12) \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}. && (13) \end{aligned}$$

Problem 10 (Newton's method of tangents for finding a root (scalar case)) We are given

$$g(x) = x^2 - 2x + 2 = 0 \qquad x^{(0)} = 2. \tag{14}$$

The first step of the algorithm yields

$$x^{(1)} = x^{(0)} - \frac{g(x^{(0)})}{g'(x^{(0)})} = 2 - 2/2 = 1. \tag{15}$$

Problem 11 (Secant method for finding a root (scalar case))

We are given

$$g(x) = x^2 - 2x + 9/4 = 0 \qquad \text{with} \qquad x^{(-1)} = 2,\ x^{(0)} = 3/2. \tag{16}$$

The first step of the algorithm yields

$$\begin{aligned} x^{(1)} &= x^{(0)} - \frac{g(x^{(0)})\left(x^{(0)} - x^{(-1)}\right)}{g(x^{(0)}) - g(x^{(-1)})} && (17)\\ &= \frac{3}{2} - \frac{(9/4 - 3 + 9/4)(3/2 - 2)}{(9/4 - 3 + 9/4) - (4 - 4 + 9/4)} && (18)\\ &= 1/2. && (19) \end{aligned}$$

Problem 12 (Newton's method for finding a minimizer (scalar case))

We are given

$$f(x) = x^2 - 2x + 2 \qquad \text{with} \qquad x^{(0)} = 3/2, \tag{20}$$

so

$$\begin{aligned} f'(x) &= 2x - 2 && (21)\\ f''(x) &= 2. && (22) \end{aligned}$$

Thus $f(x^{(0)}) = 5/4$, $f'(x^{(0)}) = 1$, and $f''(x^{(0)}) = 2$ and

$$x^{(1)} = x^{(0)} - \frac{f'(x^{(0)})}{f''(x^{(0)})} = 3/2 - 1/2 = 1. \tag{23}$$