Name: Score: /100

This exam is closed-book.

- You must show ALL of your work for full credit.
  - Please read the questions carefully.
  - Please check your answers carefully.
- Calculators may NOT be used.
  - Please leave fractions as fractions, but simplify them, etc.
  - I do not want the decimal equivalents.
- Cell phones and other electronic communication devices must be turned off and stowed along with your backpack at the front to f the room.
- Please do not write on the backs of the exam or additional pages.
  - The instructor will grade only one side of each page.
  - Extra paper is available from the instructor.
- Please write your name on every page that you would like graded.

1	2	3	4	5	6

1. (20 points) Consider the n-link rigid robot dynamic equation

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = u.$$

Identify by name and formula each variable in the above dynamic equation.

Solution:

The vector q is the vector of generalized torques.

The matrix $M(q)$ is the sum of the inertia matrix $D(q)$ and the diagonal matrix whose kth diagonal element is $r_k^2 J_{m_k}$, r_k being the gear ratio and J_{m_k} being the sum of the actuator and gear inertias for the kth link. The inertia matrix $D(q)$ is

$$D(q) = \sum_{i=1}^{n} \left(m_i J_{v_i}^T(q) J_{v_i}(q) + J_{\omega_i}^T(q) R_i(q) I_i R_i^T(q) J_{\omega_i}(q) \right)$$

where

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix},$$

the ith column of J_{v_i} is

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases}$$

and

$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}.$$

The matrix $C(q,\dot{q})$ is obtained from the Christoffel symbols according to

$$c_{ij} = \sum_{k=1}^{n} c_{kji}(q)\dot{q}_k = \left(\frac{1}{2}\right) \sum_{k=1}^{n} \left(\frac{\partial d_{ij}}{\partial q_k} + \frac{\partial d_{ik}}{\partial q_j} - \frac{\partial d_{kj}}{\partial q_i} \right) \dot{q}_k,$$

When the Euler-Lagrange equations are written as

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_k + \sum_{i,j} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k \qquad k = 1, 2, \ldots, n,$$

the terms involving $\dot{q}_i^2$ are called centrifugal and those involving $\dot{q}_i\dot{q}_j, \;\; j \neq i$ are called the Coriolis terms.

The B matrix models friction.

The gravity terms are $g_k = \frac{\partial P}{\partial q_k}$ where P is the total potential energy.

The control input u has the dimension of torque.

2. (20 points) Consider again the n-link rigid robot dynamic equation.

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = u.$$

You wish to design a joint space inverse dynamics controller. Write the expressions for the inner-loop control u and the outer-loop control a_q.

The inner-loop control is chosen to cancel the dynamics as

$$u = M(q)a_q + C(q,\dot{q})\dot{q} + B\dot{q}g(q)$$

and the outer-loop control calculates the desired acceleration. For the double integrator system, the desired acceleration is

$$a_q = \ddot{\tilde{q}}^d - K_0\tilde{q} - K_1\dot{\tilde{q}}$$

where the control error is $\tilde{q} = q - q^d$, K_0 is a diagonal matrix having ith diagonal element ω_i^2, and K_1 is a diagonal matrix having ith diagonal element $2\omega_i$. These are chosen to obtain critically damped second order behavior in each joint.

3. (25 points) Consider the two-link planar system consisting of a revolute joint rotating about a pivot mounted on a horizontal track. Derive the dynamic equations for this system.

4. (10 points) Define skew symmetry and show that

$$N(q,\dot{q}) = \dot{D}(q) - 2C(q,\dot{q})$$

is skew symmetric.

Solution:

A matrix A is skew symmetric if $a_{ij} = -a_{ji}$, which of necessity, means that the diagonal elements are zero.

To show this, we consider the element

$$n_{ij} = \dot{d}_{ij} - 2c_{ij}.$$

Taking the derivative and substituting for c_{ij}, we have

$$\begin{aligned} n_{ij} &= \dot{d}_{ij} - 2c_{ij} \\ &= \sum_{k=1}^{n} \frac{\partial d_{ij}}{\partial q_k}\dot{q}_k - 2\left(\frac{1}{2}\right)\sum_{k=1}^{n}\left(\frac{\partial d_{ij}}{\partial q_k} + \frac{\partial d_{ik}}{\partial q_j} - \frac{\partial d_{kj}}{\partial q_i}\right)\dot{q}_k \\ &= \sum_{k=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i} - \frac{\partial d_{ik}}{\partial q_j}\right)\dot{q}_k. \end{aligned}$$

Examining each term of the sum, we find that

$$\frac{\partial d_{kj}}{\partial q_i} - \frac{\partial d_{ik}}{\partial q_j} = -\left(\frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{jk}}{\partial q_i}\right)$$

so $n_{ij} = -n_{ji}$ and the matrix $N(q,\dot{q})$ is skew symmetric.

5. (20 points) Show that the two-link revolute joint arm with remotely driven link is linear in parameters. The manipulator has the kinetic energy

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

where

$$D(q) = \begin{bmatrix} m_1\ell_{c_1}^2 + m_2\ell_1^2 + I_1 & m_2\ell_1\ell_{c_2}\cos(q_2 - q_1) \\ m_2\ell_1\ell_{c_2}\cos(q_2 - q_1) & m_2\ell_{c_2}^2 + I_2 \end{bmatrix},$$

$$C(q, \dot{q}) = \begin{bmatrix} 0 & m_2\ell_1\ell_{c_2}\sin(q_2 - q_1) \\ -m_2\ell_1\ell_{c_2}\sin(q_2 - q_1) & 0 \end{bmatrix},$$

and

$$P = m_1 g\ell_{c_1}\sin q_1 + m_2 g\left(\ell_1 \sin q_1 + \ell_{c_2}\sin(q_1 + q_2)\right)$$

Find Θ and $Y(q, \dot{q}, \ddot{q})$.

First we need to find $g(q)$, which is

$$g(q) = \begin{bmatrix} \frac{\partial P}{\partial q_1} \\ \frac{\partial P}{\partial q_2} \end{bmatrix} = \begin{bmatrix} (m_1 g\ell_{c_1} + m_2 g\ell_1)\cos q_1 + m_2 g\ell_{c_2}\cos(q_1 + q_2) \\ m_2 g\ell_{c_2}\cos(q_2 + q_1) \end{bmatrix}.$$

Now we see that a minimal parameter vector is

$$\Theta = \begin{bmatrix} m_1\ell_{c_1}^2 + m_2\ell_1^2 + I_1 \\ m_2\ell_1\ell_{c_2} \\ m_2\ell_{c_2}^2 + I_2 \\ m_1 g\ell_{c_1} + m_2 g\ell_1 \\ m_2 g\ell_{c_2} \end{bmatrix}.$$

Of course the order of the Θ_is is arbitrary.

It will be more convenient to work with the equations if we separate them to obtain

$$\Theta_1\ddot{q}_1 + \Theta_2\ddot{q}_2 + \Theta_2\sin(q_2 - q_1)\dot{q}_2 + \Theta_4\cos q_1 + \Theta_5\cos(q_1 + q_2) = \tau_1$$

and

$$\Theta_2 \cos(q_2 - q_1)\ddot{q}_1 + \Theta_3 \ddot{q}_2 - \Theta_2 \sin(q_2 - q_1)\dot{q}_1 + \Theta_5 \cos(q_2 + q_1) = \tau_2.$$

Thus we have

$$Y(q, \dot{q}, \ddot{q}) = \begin{bmatrix} \ddot{q}_1 & \cos(q_2 - q_1)\ddot{q}_2 + \sin(q_2 - q_1)\dot{q}_2 & 0 & \cos q_1 & \cos(q_1 + q_2) \\ 0 & \cos(q_2 - q_1)\ddot{q}_1 - \sin(q_2 - q_1)\dot{q}_1 & \ddot{q}_2 & 0 & \cos(q_1 + q_2) \end{bmatrix}$$

6. (5 points) State the Euler-Lagrange equation for an n-DOF system, and define all of the variables in it, giving formulas where appropriate.

`Solution:`

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k$$

where q_k are the generalized coordinates,
τ_k are the corresponding generalized forces,
and $\mathcal{L} = \mathcal{K} - \mathcal{P}$ is the Lagrangian, the difference between the kinetic energy $\mathcal{K}$ and the potential energy $\mathcal{P}$.

Formula Sheet

A set of **Basic Homogeneous Transformations** that generate $SE(3)$

$$\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the **Denavit-Hartenberg (DH) convention**, A_i is the product of four basic transformations,

$$\begin{aligned} A_i &= \text{Rot}_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}\text{Rot}_{x,\alpha_i} \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$