Name: **Solution** Score: /100

This exam is closed-book.

- You must show ALL of your work for full credit.
  - Please read the questions carefully.
  - Please check your answers carefully.
- Calculators may NOT be used.
  - Please leave fractions as fractions, but simplify them, etc.
  - I do not want the decimal equivalents.
- Cell phones and other electronic communication devices must be turned off and stowed along with your backpack at the front to f the room.
- Please do not write on the backs of the exam or additional pages.
  - The instructor will grade only one side of each page.
  - Extra paper is available from the instructor.
- Please write your name on every page that you would like graded.

1	2	3	4	5	6	7	8

1. (15 points) Find $\dot{p}^0$, the linear velocity with respect to the base frame, of a point $p^1 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T$ attached to a moving frame F_1, if the moving frame is rotating at angular velocity $\omega = \hat{k}$ rad/s and translating along the x_0 axis at a rate of 1 m/s.

Solution:

Starting with the expression for p^0, we take the time derivative, applying the product rule, then note that p is fixed to Frame 1 so $\dot{p}^1 = 0$.

$$\begin{aligned} p^0 &= R_1^0 p^1 + o_1 \\ \dot{p}^0 &= \dot{R}_1^0 + R_1^0 \dot{p}^1 + \dot{o}_1 \\ \dot{p}^0 &= \dot{R}_1^0 + \dot{o}_1 \end{aligned}$$

We are given $p^1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, and $\dot{o} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

We can find $\dot{R}_1^0$ either directly or by premultiplying R_1^0 by $S(\hat{k})$. Rotation about the z axis corresponds to

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Taking the time derivative element by element yields

$$\dot{R}_1^0 = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned}
\dot{p}^0 &= \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 - 2\cos\theta \\ -2\sin\theta \\ 0 \end{bmatrix}.
\end{aligned}$$

Alternatively, since the rotation is about the k axis,

$$\begin{aligned}
\dot{p}^0 &= \dot{R}_1^0 p^1 + \dot{o}_1 \\
&= S(\hat{k}) R_1^0 p^1 \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2\sin\theta \\ 2\cos\theta \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 - 2\cos\theta \\ -2\sin\theta \\ 0 \end{bmatrix}.
\end{aligned}$$

2. (25 points) Consider the manipulator whose DH parameters are given in the table below. Compute the geometric Jacobian.

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	a_2	0	d_2^*	0
3	0	0	0	θ_3^*
4	a_4	0	d_4^*	0

`Solution:`

`First we use the table to find the` A_i`.`

$$A_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & a_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

`Next the products` A_1`,` A_1A_2`,` $A_1A_2A_3$`, and` $A_1A_2A_3A_4$`. In order to fit these on the sheet, I'll write` c_i `for` $\cos\theta_i$ `and` s_i `for` $\sin\theta_i$`.`

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_1A_2 = \begin{bmatrix} c_1 & -s_1 & 0 & a_2\,c_1 \\ s_1 & c_1 & 0 & a_2\,s_1 \\ 0 & 0 & 1 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_1A_2A_3 = \begin{bmatrix} c_1\,c_3 - s_1\,s_3 & -c_1\,s_3 - c_3\,s_1 & 0 & a_2\,c_1 \\ c_1\,s_3 + c_3\,s_1 & c_1\,c_3 - s_1\,s_3 & 0 & a_2\,s_1 \\ 0 & 0 & 1 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_1A_2A_3A_4 = \begin{bmatrix} c_1\,c_3 - s_1\,s_3 & -c_1\,s_3 - c_3\,s_1 & 0 & a_4\,(c_1\,c_3 - s_1\,s_3) + a_2\,c_1 \\ c_1\,s_3 + c_3\,s_1 & c_1\,c_3 - s_1\,s_3 & 0 & a_4\,(c_1\,s_3 + c_3\,s_1) + a_2\,s_1 \\ 0 & 0 & 1 & d_1 + d_2 + d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From these we find the frame origins o_i and the z_i. The o_i are the first three entries of the last column of the products and the z_i are the first three entries of the third column of the respective products.

o_0 is the origin of the fixed frame and the remaining o_i are

$$o_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix},\ \ o_2 = \begin{bmatrix} a_2c_1 \\ a_2s_1 \\ d_1 + d_2 \end{bmatrix},\ \ o_3 = \begin{bmatrix} a_2c_1 \\ a_2s_1 \\ d_1 + d_2 \end{bmatrix},\ \ o_4 = \begin{bmatrix} a_4(c_1c_3 - s_1s_3) + a_2c_1 \\ a_4(c_1s_3 + s_1c_3) + a_2s_1 \\ d_1 + d_2 + d_4 \end{bmatrix}.$$

The z_i are all equal to $z_0 = \hat{k}$.

Then using the appropriate formulae for the columns of the

J_v `and` J_ω `we have`

$$\begin{aligned} J_{v_1} &= z_0 \times (o_4 - o_0) \\ J_{v_2} &= z_1 \\ J_{v_3} &= z_2 \times (o_4 - o_2) \\ J_{v_1} &= z_3 \end{aligned}$$

`and`

$$\begin{aligned} J_{\omega_1} &= z_0 \\ J_{\omega_2} &= 0 \\ J_{\omega_3} &= z_2 \\ J_{\omega_4} &= 0 \end{aligned}$$

Putting all of this together yields

$$J = \begin{bmatrix} -a_4\,(c_1\,s_3 + c_3\,s_1) - a_2\,s_1 & 0 & -a_4\,(c_1\,s_3 + c_3\,s_1) & 0 \\ a_4\,(c_1\,c_3 - s_1\,s_3) + a_2\,c_1 & 0 & a_4\,(c_1\,c_3 - s_1\,s_3) & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

3. (5 points) Determine, based on the expression for the Jacobian that you found in the previous problem, whether singularities will occur. It is sufficient to give an equation in the appropriate variables (and only the appropriate variables). Solving for them is not necessary.

Solution:

Singularities occur when the Jacobian does not have full rank. If the Jacobian is square, equate its determinant to zero and solve for the locations of the singularities. Since our Jacobian is not square, it does not have full rank. Moreover, the second and fourth columns are identical, so the rank is at most 3. The first and second columns clearly form a linearly independent set. The first and third columns differ by

$$x = \begin{bmatrix} a_2 c_1 \\ a_2 s_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

which is nonzero so long as $a_2 \neq 0$, so as long as $a_2 \neq 0$, the Jacobian has rank three.

The rank deficiency due to the fourth column being identical to the second corresponds to the kernel of J being

$$\{q : q_1 = q_3 = 0, q_4 = -q_2\}.$$

This makes sense because the second and fourth joints are both prismatic, so the action of one could be compensated by the other.

4. (10 points) Design a cubic trajectory that goes from $q(t_0) = 1$ to $q(t_1) = 3$ with $\ddot{q}(t_0) = 0$, and $\dot{q}(t_0) = 0$, where $t_0 = 0$ and $t_1 = 1$ by constructing and solving the appropriate matrix equation.

Solution:

A cubic trajectory can be parametrized as

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

We are given initial conditions on $q(t)$, $\dot{q}(t)$, and $\ddot{q}(t)$, so we start by calculating the derivatives.

$$\begin{aligned} q(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3. \\ \dot{q}(t) &= a_1 + 2a_2 t + 3a_3 t^2 \\ \ddot{q}(t) &= 2a_2 + 6a_3 t. \end{aligned}$$

We can then write the matrix equation

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 0 & 0 & 2 & 6t_0 \\ 1 & t_1 & t_1^2 & t_1^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q(t_0) \\ \dot{q}(t_0) \\ \ddot{q}(t_0) \\ q(t_1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

Substituting for t_0 and t_1 and verifying that the resulting matrix is full rank[1], we can solve for the coefficients

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

As an alternative to inverting the matrix we can solve

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

by rows. From row 1 we have $a_0 = 1$. From row 2 we have $a_1 = 0$. From row 3 we have $a_2 = 0$. Then from row 4 we have $a_0 + a_3 = 3$ so $a_3 = 2$.

Thus

$$q(t) = 1 + 2t^3.$$

[1]Verification that the matrix is invertible:

$$\left| \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right| = 1 \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right| = 1(1) \left| \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \right| = 2 \neq 0.$$

5. (25 points) Consider a spherical wrist consisting of three links. If the first rotates about x_0 having $\theta_1(t) = 5t$, the second rotates about y_1 with angle $\theta_2(t) = 3t$, and the third rotates with angle $\theta_3(t) = 2t$ about the fixed z_0 axis, what is the angular velocity of the third with respective to the fixed frame, expressed in the coordinates of the fixed frame. (Hint: Don't just try to plug this into a memorized formula. Either derive the correct formula or take care to find the correct rotation matrices and angular velocities.)

Solution:

The first rotation is expressed with respect to the fixed frame. The second with respect to the first, and the third with respect to the fixed. Accordingly, the resulting rotation matrix is

$$R = R_3 R_1 R_2,$$

where

$$\begin{aligned} R_1 := R_1^0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \\ R_2 := R_2^1 &= \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}, \\ R_3 &= \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Taking the time derivative of R yields

$$\begin{aligned}
\dot{R} &= \dot{R}_3 R_1 R_2 + R_3 \dot{R}_1 R_2 + R_3 R_1 \dot{R}_2 \\
&= S(2\hat{k}) R_3 R_1 R_2 + R_3 S(5\hat{\imath}) R_1 R_2 + R_3 R_1 S(3\hat{\jmath}) R_2 \\
&= S(2\hat{k}) R_3 R_1 R_2 + R_3 S(5\hat{\imath}) R_3^T R_3 R_1 R_2 + R_3 R_1 S(3\hat{\jmath}) R_1^T R_1 R_2 \\
&= S(2\hat{k}) R + S(R_3 5\hat{\imath}) R_3 R_1 R_2 + R_3 S(R_1 3\hat{\jmath}) R_1 R_2 \\
&= S(2\hat{k}) R + S(R_3 5\hat{\imath}) R + R_3 S(R_1 3\hat{\jmath}) R_3^T R_3 R_1 R_2 \\
&= S(2\hat{k} + R_3 5\hat{\imath}) R + S(R_3 R_1 3\hat{\jmath}) R \\
&= S(2\hat{k} + R_3 5\hat{\imath} + R_3 R_1 3\hat{\jmath}) R.
\end{aligned}$$

The second term of the angular velocity is

$$R_3 5\hat{\imath} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5\cos\theta_3 \\ 5\sin\theta_3 \\ 0 \end{bmatrix}.$$

The third term is

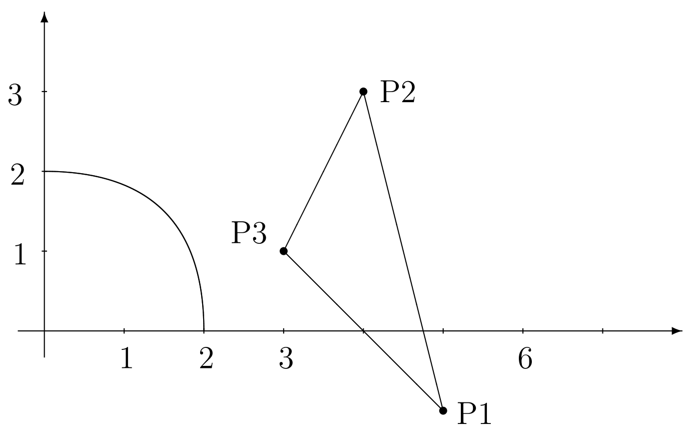
$$\begin{aligned}
R_3 R_1 3\hat{\jmath} &= \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3\cos\theta_1 \\ 3\sin\theta_1 \end{bmatrix} \\
&= \begin{bmatrix} -3\cos\theta_1 \sin\theta_3 \\ 3\cos\theta_1 \cos\theta_3 \\ 3\sin\theta_1 \end{bmatrix}.
\end{aligned}$$

Thus the angular velocity is

$$\begin{aligned}
\omega &= 2\hat{k} + R_3 5\hat{\imath} + R_3 R_1 3\hat{\jmath} \\
&= 2\hat{k} + \left(5\cos\theta_3 \hat{\imath} + 5\sin\theta_3 \hat{\jmath}\right) + R_3 R_1 3\hat{\jmath} \\
&= 2\hat{k} + \left(5\cos\theta_3 \hat{\imath} + 5\sin\theta_3 \hat{\jmath}\right) + \left(-3\cos\theta_1 \sin\theta_3 \hat{\imath} + 3\cos\theta_1 \cos\theta_3 \hat{\jmath} + 3\sin\theta_1 \hat{k}\right) \\
&= \left(5\cos\theta_3 - 3\cos\theta_1 \sin\theta_3\right)\hat{\imath} + \left(5\sin\theta_3 + 3\cos\theta_1 \cos\theta_3\right)\hat{\jmath} + \left(2 + 3\sin\theta_1\right)\hat{k}.
\end{aligned}$$

6. (5 points) A robotic vacuum cleaner of radius 2 is centered at location $(0,0)$ in a room. It must avoid a triangular object with corners at $P_1(5,-1)$, $P_2(4,3)$, and $P_3(3,1)$. Find the shortest distance to each side of the triangular object.

`Solution: Drawing the objects is always a good idea.`



The triangular object is bounded by three line segments: $\overline{P_1P2}, \overline{P2P3}$, and $\overline{P3P1}$. They have slopes

$$\begin{aligned} m_{12} &= (y_2 - y_1)/(x_2 - x_1) = (3-(-1))/(4-5) = -4, \\ m_{23} &= (y_3 - y_2)/(x_3 - x_2) = (1-3)/(3-4) = 2, \\ m_{31} &= (y_1 - y_3)/(x_1 - x_3) = (-1-1)/(5-3) = -1. \end{aligned}$$

The intercepts corresponding to the lines through containing these line segments are:

$$\begin{aligned} b_{12} &= y_1 - m_{12}x_1 = -1 + 4(5) = 19, \\ b_{23} &= y_2 - m_{23}x_2 = 3 - 2(4) = -5, \\ b_{31} &= y_3 - m_{31}x_3 = 1 + 1(3) = 4. \end{aligned}$$

The perpendicular to $\overline{P_1P_2}$ then has slope $-(-1/4) = 1/4$. The perpendicular to $\overline{P_1P_2}$ has slope $-1/2$, and the perpendicular to $\overline{P_1P_2}$ has slope $-(-1) = 1$.

Solving for the locations (x_i^*, y_i^*) of the intersections yields the following equations:

$$\begin{aligned} y_1^* &= -4x_1^* + 19 = x_1^*/4, \\ y_2^* &= 2x_2^* - 5 = -x_2^*/2, \\ y_3^* &= -x_3^* + 4 = x_3^*. \end{aligned}$$

Solving, we obtain intersection points $(19(4)/17, 19/17)$, $(2, -1)$, and $(2, 2)$, respectively. Only the first of these lies on its corresponding line segment. Thus the closest point on $\overline{P_2P_3}$ and on $\overline{P_3P_1}$ is their intersection P_3. The distance from the robot to P_3 is $\sqrt{3^2+1^2} - 2 = \sqrt{10} - 2$. (The robot has radius 2.) Checking that the robot and triangular object do not overlap, $\sqrt{10} > 3$ so the distance is positive.

The distance to the line segment $\overline{P_1P_2}$ is $19\sqrt{17}/17 - 2$. Again checking our answer, overlap will not occur as this point is farther from the robot than P_3 is, and indeed, $19/17 > 1$ and $\sqrt{17} > 2$ so the distance is positive.

7. (5 points) Give the formula for the workspace repulsive force that should be used to keep the robot from hitting the triangular object. If $\eta_i = 1, \forall i$ and $\rho_0 = 1$, find the repulsive force corresponding to the closest point on the triangular object. You need not calculate a final value, but should supply all values required by the formula.

`Solution:`

`We use the repulsive potential field`

$$U_{rep,i}(q) = \begin{cases} \frac{1}{2}\left(\frac{1}{\rho(o_i(q))} - \frac{1}{\rho_0}\right)^2 & \rho(o_i(q)) \leq \rho_0 \\ 0 & \rho(o_i(q)) > \rho_0. \end{cases}$$

`The repulsive force is the gradient of the potential field`

$$F_{rep,i}(q) = \begin{cases} \eta_i\left(\frac{1}{\rho(o_i(q))} - \frac{1}{\rho_0}\right)\frac{1}{\rho^2(o_i(q))}\nabla\rho(o_i(q)) & \rho(o_i(q)) \leq \rho_0 \\ 0 & \rho(o_i(q)) > \rho_0, \end{cases}$$

`where` $\rho_i(o_i(q))$ `is the shortest distance from` $o_i(q)$ `to any workspace obstacle.`

`Let` b `be the point of the object closest to` $o_i(q)$`. Since the obstacle is convex, the gradient is`

$$\nabla\rho(x)\big|_{x=o_i(q)} = \frac{o_i(q) - b}{\|o_i(q) - b\|},$$

`the unit vector directed from` b `toward` $o_i(q)$`.`

8. (10 points) State the Euler-Lagrange equation for an n-DOF system, and define all of the variables in it, giving formulas where appropriate.

Solution:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k$$

where q_k are the generalized coordinates, τ_k are the corresponding generalized forces, and $\mathcal{L} = \mathcal{K} - \mathcal{P}$ is the Lagrangian, the difference between the kinetic energy $\mathcal{K}$ and the potential energy $\mathcal{P}$.

Formula Sheet

A set of **Basic Homogeneous Transformations** that generate $SE(3)$

$$\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the **Denavit-Hartenberg (DH) convention**, A_i is the product of four basic transformations,

$$\begin{aligned} A_i &= \text{Rot}_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}\text{Rot}_{x,\alpha_i} \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$