Name: Score: /100

This exam is closed-book.

- You must show ALL of your work for full credit.
  - Please read the questions carefully.
  - Please check your answers carefully.
- Calculators may NOT be used.
  - Please leave fractions as fractions, but simplify them, etc.
  - I do not want the decimal equivalents.
- Cell phones and other electronic communication devices must be turned off and stowed under your desk.
- Please do not write on the backs of the exam or additional pages.
  - The instructor will grade only one side of each page.
  - Extra paper is available from the instructor.
- Please write your name on every page that you would like graded.

1	2	3	4	5	6	7	8	9	10

1. (5 points) Starting from Frame F_0, rotate first by θ_1 about the fixed z-axis and then by θ_2 about the fixed x-axis to orient Frame F_1. Find the rotation matrix representing this transformation.

2. (5 points) Starting from Frame F_0, rotate first by θ_1 about the fixed z-axis and then by θ_2 about the current x-axis to orient Frame F_1. Find the rotation matrix representing this transformation.

3. First rotate by $\pi/2$ about the fixed y-axis. Call the new frame F_1. Then translate 1 cm along the current z-axis. Call the new frame F_2. Finally translate 2 cm along the fixed x-axis. Call the new frame F_3.

   (a) (10 points) Find the H matrix corresponding to this transformation.

   (b) (5 points) Now reverse the order of operations and find the matrix corresponding to this new transformation.

   (c) (5 points) What is the origin of F_3 with respect to the fixed frame?

4. (10 points) Consider a point p having coordinates $p^2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$. Using the H you just found in part (a) of problem 3, determine the following.

   (a) What are the coordinates of p with respect to the fixed frame?

   (b) What are the coordinates of p with respect to frame F_3?

5. (5 points) If link Li has link twist 0, link length 2, and link offset 4, and joint Ji is positioned at joint angle $\pi/2$, what is A_i?

6. (10 points) Identify the DH parameters for the matrix

$$A_i = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{3\sqrt{3}}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Assume that all angle measurements are between 0 and 2π.

7. (10 points) What is the origin of frame F_0 with respect to frame F_1 if

$$T_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 3 \\ 0 & 1 & 0 & 4 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}?$$

8. (15 points) Give the expressions for T^i_j, in terms of the appropriate A_i, for all possible pairs i and j.

9. (15 points) Consider the vectors $a = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}^T$, $p = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$, and the rotation matrix R corresponds to a rotation of π radians about the z-axis.

   (a) What is $S(a)$?

   (b) Use skew symmetric matrices to find $a \times p$.

   (c) Use skew symmetric matrices to find $RS(a)R^T$.

10. (5 points) Use skew-symmetric matrices to find

$$\frac{d}{d\alpha}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}\right).$$

Formula Sheet

A set of **Basic Homogeneous Transformations** that generate $SE(3)$

$$\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the **Denavit-Hartenberg (DH) convention**, A_i is the product of four basic transformations,

$$\begin{aligned} A_i &= \text{Rot}_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}\text{Rot}_{x,\alpha_i} \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$