A **proposition** is any declarative sentence (including mathematical sentences such as equations) that is true or false.

Example: *Snow is white* is a typical example of a proposition. Most people would agree that it's a true one, but in the real world few things are absolute: city dwellers will tell you that snow can be grey, black, or yellow.

Example: *3 + 2 = 5* is a simple mathematical proposition. Under the most common interpretation of the symbols in it, it is of course true.

Example: *3 + 2 = 7* is also a proposition, even though it is false in the standard number system. Nothing says a proposition can't be false. Also, this equation could be true (and the previous one false) in a nonstandard number system.

Example: *Is anybody home?* is not a proposition; questions are not declarative sentences.

We use the letters $P, Q, R, \ldots$ as propositional variables.

- these letters stand for or represent statements, in much the same way that a mathematical variable like x represents a number.

Logic connectives connectives, are used to stand for the following words:

- $\wedge$ “and”
- $\vee$ ”or”
- $\sim$ ”not”
- $\rightarrow$ ”implies” or ”if ... then”
- $\leftrightarrow$ ”if and only if”
- also use parentheses ( ) for grouping

The words themselves, as well as the symbols, may be called connectives. Using the connectives, we can build new statements from simpler ones. Specifically, if P and Q are any two statements, then

$P \wedge Q$,

$P \vee Q$,

$\sim P$,

$P \rightarrow Q$,

$P \leftrightarrow Q$

A statement that is not built up from simpler ones by connectives and/or quantifiers is called atomic or simple.

A statement that is built up from simpler ones is called compound.

The truth functions of the connectives are defined as follows:

$P \wedge Q$, is true provided both P and Q are true

$P \vee Q$, is true provided at least one of P or Q is true

$\sim P$, is true provided P is true

$P \rightarrow Q$, is true as long as we don't have P is true and Q is false

$P \leftrightarrow Q$ is true provided P and Q are the same (either both true or both false)

Truth tables of the connectives

Definition: A **tautology**, or a law of propositional logic, is a statement which is always true

A **contradiction** is a statement whose truth function has all Fs as outputs (in other words, it's a statement whose negation is a tautology).

Two statements are called propositionally equivalent if a tautology results when the connective $\sim$ is placed between them.

Example: One simple tautology is the symbolic statement $P \to P$. This could represent an English sentence like
"If I don't finish, then I don't finish."
Note that this sentence is obviously true, but it doesn't convey any information. This is typically the case with such simple tautologies.

One of the simplest and most important contradictions is the statement $P \wedge \sim P$.
An English example would be
"I love you and I don't love you."
Although this statement might make sense in a psychological or emotional context, it is still a contradiction. That is, from a logical standpoint it cannot be true.

The statement $\sim P \to Q$ is propositionally equivalent to $P \vee Q$,
For instance, if I say, "If I don't finish this chapter this week, I'm in trouble,"
this is equivalent to saying (and so has essentially the same meaning as),
"I (must) finish this chapter this week or I'm in trouble."

Truth Table of $P \wedge Q) \vee \sim P$

Truth Table of $P \rightarrow [Q \rightarrow (P \wedge Q)]$

Truth Table of $(P \rightarrow Q) \leftrightarrow (R \wedge P)$

Example: For each of the following statements, introduce a propositional variable for each of its atomic sub statements, and then use these variables and connectives to write the most accurate symbolic translation of the original statement.

- I like milk and cheese but not yogurt.
- Rain means no soccer practice.
- The only number that is neither positive nor negative is zero.
- 2+2=4

In any conditional $P \to Q$, the statement P is called the hypothesis or antecedent and Q is called the conclusion or consequent of the conditional.

Definitions: Given any conditional $P \to Q$,
the statement $Q \to P$ is called its converse.
the statement $\sim P \to \sim Q$ is called its inverse.
the statement $\sim Q \to \sim P$ is called its contrapositive.

We now come to the first result in this text that is labeled a "theorem."

Theorem (a) Every conditional is equivalent to its own contrapositive.
(b) A conditional is not necessarily equivalent to its converse or its inverse.
(c) However, the converse and the inverse of any conditional are equivalent to
each other.
(d) The conjunction of any conditional $P \to Q$ and its converse is equivalent
to the biconditional $P \leftrightarrow Q$.

Proof

Example: Consider the conditional "If you live in California, you live in America." This statement is true for all persons.

Its converse is "If you live in America, you live in California";

its inverse is "If you don't live in California, you don't live in America."

These two statements are not true in general, so they are not equivalent to the original. However, they are equivalent to each other.

The contrapositive of the original statement is "If you don't live in America, you don't live in California," which has the same meaning as the original and is always true.

When we say "P implies Q" or even "If P then Q," we normally mean that the statement P, if true, somehow causes or forces the statement Q to be true. In mathematics, most conditionals convey this kind of causality, but it is not a requirement.

In logic (and therefore in mathematics), the truth or falsity of a conditional is based strictly on truth values.

Example: The following three statements, although they may seem silly or even wrong, must be considered true:
If $2 + 2 = 4$, then ice is cold.
If $2 + 2 = 3$, then ice is cold.
If $2 + 2 = 3$, then ice is hot.
On the other hand, the statement "If $2 + 2 = 4$, then ice is hot" is certainly false.

The most common ways to express a conditional $P \rightarrow Q$ in words

- P implies Q.
- If P then Q.
- If P, Q.
- Q if P.
- P only if Q.
- P is sufficient for Q
- Q is necessary for P.
- Whenever P, Q.
- Q whenever P

The most common ways to express a biconditional $P \leftrightarrow Q$ in words

- P if and only if Q.
- P is necessary and sufficient for Q.
- P is equivalent to Q.
- P and Q are equivalent.
- P (is true) just in case Q (is).

Example: Prove “ The sum of two even numbers must be even.”

Propositional Consequence; Introduction to Proofs

to be continued

Propositional Consequence; Introduction to Proofs

the concepts of tautology and propositional equivalence have been introduced

because we can build on the logic of tautolgies and derive other consequences

for example

```
Suppose we have
P --> Q is true
Now suppose we know P is true
then we can deduce Q
```

A statement Q is said to be a propositional consequence of statements $P_1, P_2, \ldots, P_n$ iff the single statement $P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$ is a tautology.

The assertion that a statement Q is a consequence of some list of statements is called an **argument**.

The statements in the list are called the **premises** or **hypotheses** of the argument, and Q is called the **conclusion** of the argument. If Q really is a consequence of the list of statements, the argument is said to be valid.

```
Premises:   P --> Q
            P
-----------------
conclusion  Q


Premises:   P --> Q
            ~R --> ~Q
            ~R
-----------------
conclusion   ~P
```


see table 2.5

Premises: If I'm right, you're wrong. If you're right, I'm wrong.
Conclusion: Therefore, at least one of us is right.

If Al shows up, Betty won't. If Al and Cathy show up, then so will Dave. Betty or Cathy (or both) will show up. But Al and Dave won't both show up. Therefore, Al won't show up.

Theorem Given sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Let $x \in A$...........complete in class.....

Predicate Logic

A mathematical variable is a symbol (or combination of symbols like x) that stands for an unspecified number or other object.

The collection of objects from which any particular variable can take its values is called the *domain* or the *universe* of that variable

Example: If you saw the equation $f(x) = 3$, you would probably read this as "f of x equals 3," because you recognize this as an example of function notation.

You would probably also think of x as the only variable in this equation. But strictly speaking, this equation contains two variables: x, presumably standing for a number, and f, presumably standing for a function.

There is nothing that says what letters must be used to stand for what in mathematics, but there are certain conventions or traditions that most people stick to avoid unnecessary confusion.

In algebra and calculus, for example, the letters x, y, and z almost always stand for real numbers, whereas the letters f and g stand for functions.

The fact that almost everyone automatically interprets the equation $f(x) = 3$ in the same way shows how strong a cue is associated with certain letters.

The difference between propositional variables and mathematical variables is very important, and you should be careful not to confuse them.

A propositional variable always stands for a statement-spoken, written, mathematical, English, Swedish, or whatever-that could take on a value of true or false. A mathematical variable can stand for almost any type of quantity or object except a statement.

Not every letter that stands for something in mathematics is a variable.

A symbol (or a combination of symbols) that stands for a fixed number or other object is called a constant symbol or simply a constant.

One of the first things taught in grammar is that a sentence must have a verb.

This is just as true in mathematics as it is in English.

The word “equals” and “$=$” is a verb, and the word group “is less than” and “$<$” is a verb and....

mathematical symbols like $+$, -, $\cdot$ and $/$ are used to form terms that denote objects, is function symbols or operator symbols.

Quantifiers

The study of quantifiers, together with connectives and the concepts discussed in the previous section, is called predicate logic, quantifier logic, first-order logic, or the predicate calculus.

Two symbols, called quantifiers, stand for the following words:

$\forall$ for "for all" or "for every" or "for any"

$\exists$ for "there exists" or "there is" or "for some"

$\forall$ is called the universal quantifier;

$\exists$ is called the existential quantifier.

The quantifiers are used in symbolic mathematical language as follows: if P is any statement, and x is any mathematical variable (not necessarily a real number variable), then $\forall xP$ and :$\exists xP$ are also statements.

Example: Quantifiers are used in ordinary life as well as in mathematics. For example, consider the argument:
"Susan has to show up at the station some day this week at noon to get the key. So if I go there every day at noon, I'm bound to meet her."

The logical reasoning involved in this conclusion is simple enough, but it has nothing to do with connectives. Rather, it is an example of a deduction based on quantifier logic

Definitions: A mathematical variable occurring in a symbolic statement is called **free** if it is unquantified and **bound** if it is quantified.

If a statement has no free variables it's called closed.

Otherwise it is called a predicate, an open sentence, an open statement, or a propositional function.

Example: In the statement $\forall x(x^2 \geq 0)$, the variable x is bound, so the statement is closed.

In the statement $\forall x \exists y(x - y = 2z)$ x and y are bound whereas z is free.
This statement is open; it is a propositional function of z.

A free variable represents a genuine unknown quantity whose value you probably need to know to tell whether the statement is true or false.

For example, given a simple statement like "5 + x = 3," you can't determine whether it's true or false until you know the value of the free variable x.

But a bound variable is quantified; this means that the statement is not talking about a single value of that variable.

If you are asked whether the statement "$\exists x(5 + x = 3)$" is true, it wouldn't make sense to ask what the value of x is; instead, it would make sense to ask what the domain of x is

A statement of the form $\forall x P(x)$ is defined to be true provided $P(x)$ is true for each particular value of x from its domain.

Similarly, $\exists x P(x)$ is defined to be true provided $P(x)$ is true for at least one value of x from that domain .

The following rule of thumb is also helpful: The symbolic translation of a statement must have the same free variables as the original statement.

Example: For each of the following, write a completely symbolic statement of predicate logic that captures its meaning.

(a) All gorillas are mammals.

(b) Some lawyers are reasonable.

(c) No artichokes are blue.

(d) Everybody has a father and a mother.

(e) Some teachers are never satisfied.

(f) (The number) x has a cube root.

(g) For any integer greater than 1, there's a prime number strictly between it and its double.

Working with Quantifiers

Assume that x and y are real variables and consider a simple atomic statement like $x + y = 0$

One simple way to quantify this, with no alternations, is $\exists x \exists y (x + Y = 0)$.

What does this quantified statement say, and is it true or false?

Technically, the statement says that there is a value of x for which $\exists y (x + y = 0)$ is true.

But there's no need to split up the quantifiers in this way.

"There exist x and y such that x + y = 0."

Now let's consider $\exists y \forall x (x + y = 0)$.

This says that there is a value for y that makes the statement $\forall x (x + y = 0)$ hold.

That is, there would have to exist a single value of y, chosen independently of x, that makes the inner equation work for all values of x.

Theorem Suppose a statement begins with a sequence of quantifiers, followed by some inner statement with no quantifiers. Then the statement is true provided each existentially quantified variable is definable as a function of some or all of the universally quantified variables to the left of it, in a way that makes the inner statement always true.

Prove the following

Theorem: For any two real numbers, there is a real number greater than both of them

Proof: In symbols, what we want to prove is
$\forall x, y, \exists z(z > x \wedge z > y)$

Example: For each statement, determine whether it's true in each of these number systems: the set of all natural numbers (positive integers) $\mathbb{N}$, the set of all integers $\mathbb{Z}$, , the set of all real numbers $\mathbb{R}$,, and the set of all complex numbers $\mathbb{C}$,.

(a) $\forall x, y \exists z(x + z = y)$

(b) $\exists x \forall y(x < y)$

(c) $\exists x \forall y \exists z(x = y \vee yz = 1)$

A law of logic is a symbolic statement that is true for all possible interpretations of the variables, constants, predicate symbols, and operator symbols occurring in it.

That is, it must be true no matter what domains are chosen for its bound variables, no matter what values are chosen for its constants and free variables, and so on.

Only the connectives, the quantifiers, and the equal sign are not subject to reinterpretation.

A statement Q is said to be a logical consequence of a finite list of statements $P_1, P_2, \ldots, P_n$ iff the single statement $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \rightarrow Q$ is a law of logic.

Two symbolic statements are called logically equivalent provided that each of them is a logical consequence of the other.

Negations of Statements with Quantifiers

Suppose that P is a statement that begins with a sequence of quantifiers. We've said that P is true provided that certain functions and/or constants (corresponding to the existential quantifiers of P) exist.

So we could say that P is false provided that not all these functions and/or constants exist. However, often this view of the situation doesn't help to figure out whether the statement is false.

Theorem For any statement P(x):

(a) $\sim \forall x P(x)$ is logically equivalent to $\exists x \sim P(x)$.

(b) $\sim \exists x \ P(x)$ is logically equivalent to $\forall x \sim P(x)$.

Example: Simplify each of the following statements by moving negation signs inward as much as possible.

(a) $\sim \exists x, y \forall z \sim \exists u \forall w P$

(b)

$\sim \exists x \forall t [t > 0 \rightarrow \exists d (d > 0 \wedge \forall u (|x - u| < d \rightarrow |f(x) - f(y)| > t))]$

Example: Consider the statement "Everybody has a friend who is always honest"

(a) Write a symbolic translation of this statement.

(b) Write the negation of this symbolic statement and then simplify it as in the previous examples.

(c) Translate your answer to part (b) back into reasonable-sounding English.

Theorem There is no smallest positive real number.

Proof: ..discuss in class....

Uniqueness

Recall that the existential quantifier has the meaning ”there is at least one,” which makes it analogous to the ”inclusive-or” meaning of the disjunction connective. In mathematics we often want to say that there is exactly one number (or other object) satisfying a certain condition.

In mathematics, the word ”unique” is used to mean ”exactly one.” Should we introduce a third quantifier with this meaning?

There are several different-looking but equivalent ways to say that there's a unique object satisfying a certain condition.

Different Types of Proofs

An **axiom system** consists of two parts: a list of statements that are to be considered **axioms**, and a list of **rules of inference**.

The lists mentioned in this definition may be finite or infinite. But in either case, the axioms and rules of inference must be clearly and unambiguously defined, so that it's always possible to determine whether any given statement is an axiom or follows from certain other statements by a rule of inference.

A **formal proof** is a finite sequence of statements in which every statement (or step) is either (1) an axiom, (2) a previously proven statement, (3) a definition, or (4) the result of applying a rule of inference to previous steps in the proof.

A **theorem** is a statement that can be formally proved. That is, it's a statement for which there's a formal proof whose last step is that statement.

Note there are several other words with more or less the same meaning. A relatively simple theorem may be called a **proposition**.

A theorem that is not considered very important on its own but is useful for proving a more important result is usually called a **lemma**.

A theorem that is easily proved from another theorem is usually called a **corollary** to the other theorem.

There are no hard-and-fast rules for which of these words to apply to a given result.

Some important results in mathematics have been labeled propositions or lemmas,

Sometimes it is appropriate to begin a proof with one or more assumptions (also called hypotheses or premises or “givens”).

The Use of Propositional Logic in Proofs

Propositional Consequence (PC)

In a proof, you may assert any statement that is a propositional consequence of previous steps in the proof.

Example: It is a theorem of calculus that if a function is differentiable, it is continuous.

Suppose that we know this result, and we want to assert its contrapositive in a proof; that is, if a function is not continuous, then it is not differentiable.

The rule PC allows us to do this, using the simple tautology $(P \to Q) \leftrightarrow (\sim Q \to \sim P)$.

having the rule of inference PC essentially makes all tautologies axioms. We now make this explicit. All tautologies are axioms.

Modus Ponens

If you have a step P and another step of the form P -+ Q; you may then conclude the statement Q.

Example: One important theorem of calculus is that if a function is differentiable, it must be continuous.

Another basic result is the derivative formula for polynomials, which guarantees that polynomial functions are differentiable.

Applying Modus Ponens to these steps yields that any given polynomial function, such as $3x^2 - 6x + 2$, must be continuous.

Conditional Proof

If you can produce a proof of Q from the assumption P, you may conclude the single statement $P \to Q$ (without considering P an assumption of the proof).

Even though conditional proof is derivable from propositional consequence, it is so important that we have included it separately in our axiom system. Conditional proofs are often referred to as **direct proofs of implications**.

```
   Assume    P
   [....Some correct intermediate steps....]
   Q
------------------------
   P -> Q
```

Indirect Proof

If you can produce a proof of any contradiction from the assumption $\sim P$, you may conclude P.

Example: Indirect proof is the most efficient way to prove that the sum of a rational number and an irrational number must be irrational. (Recall that a rational number is one that can be written as a fraction of integers.) Here is a proof

Assume the claim is false. Then we have $a + b = c$, for some numbers $a, b,$ and c, with a and c rational and b irrational. Simply subtract a from both sides to obtain $b = c - a$. Since the difference of two fractions can always be written as a single fraction, this makes b rational, a contradiction.

Proof by Cases

If you have a step of the form $Q \vee R$ and the two implications $Q \rightarrow P$ and R -+ P, you may conclude the statement P.

```
    Q or  R (Proved somehow)
    Case 1: Assume Q
         [....Some correct intermediate steps...]
         P (End of Case 1)
    Case 2: Assume R
         [....Some correct intermediate steps...]
         P (End of Case 2)
-------------------------------------------------
    P
```

Biconditional Rule

If you have implications $P \to Q$ and $Q \to P$, you may conclude the biconditional $P \leftrightarrow Q$.

The notation S[P] denotes a statement S that contains some statement P as a substatement (which could be the whole statement S).

The notation S[P/Q] denotes a statement that results from S[P] by replacing one or more of the occurrences of the statement P within S[P] by the statement Q.

Substitution

From statements $P \leftrightarrow Q$ and $S[P]$, you may conclude $S[P/Q]$ as long as no free variable of P or Q becomes quantified in $S[P]$ or $S[P/Q]$.

Example: Here is a simple example of substitution from real life. Suppose you say, "If Harry shows up at my party, I'll call the police."

Then your friend says, “But Harry and your boss do everything together; if one shows up, so will the other.”

Then you say, “Well, I guess that means that if my boss shows up, I’ll call the police.” You have just used substitution, because the second part of your friend’s statement means,

“Harry will show up if and only if your boss does.;;

Conjunction

If you have, as separate steps, any two statements P and Q, you may conclude the single statement $P \wedge Q$. This rule of inference follows trivially from propositional consequence.

Modus Tollens

If you have a step of the form $P \to Q$ and also have $\sim Q$, you may assert $\sim P$

Contrapositive Conditional Proof

If you can produce a proof of $\sim P$ from the assumption $\sim Q$, you may conclude the single statement $P \to Q$.

We have now discussed two ways to prove an implication $P \to Q$, but are at least three common ways:

(1) Direct conditional proof: Assume P, and derive Q.

(2) Contrapositive conditional proof: Assume $\sim Q$, and derive $\sim P$

(3) Indirect proof: Assume $\sim (P \to Q)$, or equivalently assume $P \wedge \sim Q$, and derive a contradiction.

Use of Quantifiers in Proofs

De Morgan's Laws for Quantifiers

$$\sim \forall x P(x) \leftrightarrow \exists x \sim P(x)$$
$$\sim \exists x P(x) \leftrightarrow \forall x \sim P(x)$$

Axiom: Universal Specification or US

$$\forall x P(x) \rightarrow P(t)$$

where the letter t here denotes any term or expression (not necessarily a single letter like a variable or constant) that represents an object in the domain of the variable x.

Example: You have been using US ever since you first studied algebra, even if you didn't have a name for it. For instance, consider a typical algebra formula such as

$(x + y)^2. = x^2 + 2xy + y^2$

By the convention stated at the beginning of this section, the variables x and y in this formula are assumed to be universally quantified. So when you learned this formula in high school, you learned that it was true for all numbers, and that therefore you could substitute any expression for x and/or y. So you knew that you could write

$(3a + 2)^2 = 9a^2 + 12a + 4$

$(x^2 + 5y^3)^2 = x^4 + 10x^2y^3 + 25y^6$

Every time you make this type of substitution for a variable, you are using US

Universal Generalization or UG

If you can produce a proof of P(x), where x is a free variable representing an arbitrary member of a certain domain, you may then conclude $\forall x P(x)$.

Existential Specification or ES

If you have a step of the form $\exists x P(x)$, you may assert P(c), where c is some constant symbol.

Axiom: Existential Generalization or EG

$P(t) \rightarrow \exists x P(x)$

where t is a term with the same restrictions as in the rule US.

Mathematicians frequently talk about counterexamples, primarily as a method of disproving statements.

This method is just a special case of EG, but it is used so often that it deserves separate discussion.

One standard type of mathematics problem asks the reader to *prove* or *disprove* some statement.

This often involves more work than a problem that just asks the reader to prove a statement: first you have to determine (or at least guess) whether the statement is true or false; then you must prove the statement or its negation.

Now, imagine that you are asked to prove or disprove a statement of the form $\forall x P(x)$.

If you think the statement is true, you probably try to prove it by UG. But if you think it's false, how do you disprove it?

Find a specific example of an object for which P is false. Such an example is called a **counterexample** to the statement $\forall x P(x)$ and the counterexample proves $\sim \forall x P(x)$

Example: prove or disprove For all integers: the sum of two squares is a square

Reversibility

you can do anything you want to both sides of an equation, but there are some subtleties involved with this rule.

You may remember these subtleties from precalculus/algebra.

They have to do with reversibility of steps used in solving or simplifying an equation.

Mathematical Induction

Mathematical induction is different from the axioms and rules of inference described in the previous three It is not based on logic.

Technically, it is an axiom for just one particular system, the natural numbers.

It is based on the axiom of well-ordering

we will denote the natural numbers by N or by $\mathbb{N}$

Math induction is based on the following two

simple induction

$P(x_0) \wedge [\,(\forall x > x_0)P(x) \rightarrow P(x+1))\,] \rightarrow \forall(x \geq x_0)P(x)$

generalized induction

$P(x_0) \wedge [\,(\forall x > x_0)(\forall k(x_0 \leq k < x)P(k) \rightarrow P(x+1)\,] \rightarrow \forall(x \geq x_0)P(x)$

Example: use mathematical induction to prove
$1 + 2 + 3 + ... + n = n(n + 1)/2$

Example: show that positive integers are divisible by 3 iff the sum of the digits of the integer is divisible by 3

Example: $\frac{d}{dx}(x^n) = n^{xn-1}$ Use the product rule and the derivative of x