ECE 301 Solution to Homework Assignment 2

1. Indicate whether the following systems are causal, invertible, linear, memoryless, and/or time invariant by circling the correct options. (A system may have more than one of these properties.) Justify your answer.

   (a) $y(t) = x(t-2) + x(2-t)$ (causal, invertible, $\boxed{\text{linear}}$, memoryless, $\boxed{\text{time invariant}}$)

   Causality: The system is NOT causal because for any $t < 1$, the output depends on a future input, e.g. $y(0) = x(-2) + x(2)$.

   Invertibility: The system is NOT invertible because, if $w(t)$ is even, then the output $y(t)$ corresponding to $x_1(t) = w(t)u(t)$ is indistinguishable from that corresponding to $x_2(t) = w(t)u(-t)$.

   Linearity: The system is linear because if

$$\begin{aligned} y_1(t) &= x_1(t-2) + x_1(2-t), && (1)\\ y_2(t) &= x_2(t-2) + x_2(2-t), \quad \text{and} && (2)\\ x(t) &= \alpha x_1(t) + \beta x_2(t), && (3) \end{aligned}$$

   then the output $y(t)$ corresponding to the input $x(t)$ is

$$\begin{aligned} y(t) &= x(t-2) + x(2-t) && (4)\\ &= (\alpha x_1 + \beta x_2)(t-2) + (\alpha x_1 + \beta x_2)(2-t) && (5)\\ &= \alpha x_1(t-2) + \beta x_2(t-2) + \alpha x_1(2-t) + \beta x_2(2-t) && (6)\\ &= \alpha\left(x_1(t-2) + x_1(2-t)\right) + \beta\left(x_2(t-2) + x_2(2-t)\right) && (7)\\ &= \alpha y_1(t) + \beta y_2(t). && (8) \end{aligned}$$

   Memorylessness: The system is NOT memoryless because the output at time t depends on input values at times other than t.

   Time Invariance: The system is time invariant. To see this, we let $y(t)$ be the output corresponding to the input $x(t)$ and let $x_a(t) = x(t-a)$[1]. Then the output $y_a(t)$ corresponding to the input signal $x_a(t)$ is

$$\begin{aligned} y_a(t) &= x_a(t-2) + x_a(2-t) && (9)\\ &= x((t-a)-2) + x(2-(t-a)) && (10)\\ &= y(t-a). && (11) \end{aligned}$$

   (b) $y(t) = \int_{-\infty}^{2} x(\tau)d\tau$ (causal, invertible, $\boxed{\text{linear}}$, memoryless, time invariant)

[1]The *character* "a" on the left hand side of the equals sign is a label. The *variable* a on the right hand side represents an arbitrary real number. It may be easier to think of a specific number, say, 3, so long as you note that for the system to be time invariant, the test must work for any real value, not just 3.

Causality: The system is NOT causal because for any $t < 2$, the output depends on a future input.

Invertibility: The system is NOT invertible because only the integral and not the input function itself determines the output, e.g. the output corresponding to input $x_1(t) = u(t) - u(t-1)$ is the same as that corresponding to input $x_2(t) = 2[u(t) - u(t - 1/2)]$.

Linearity: The system is linear because if

$$y_1(t) = \int_{-\infty}^{2} x_1(\tau) d\tau, \text{ and} \tag{12}$$

$$y_2(t) = \int_{-\infty}^{2} x_2(\tau) d\tau, \tag{13}$$

and $x(t) = \alpha x_1(t) + \beta x_2(t)$, then the output $y(t)$ corresponding to the input $x(t)$ is

$$\begin{aligned} y(t) &= \int_{-\infty}^{2} (\alpha x_1 + \beta x_2)(\tau) d\tau && (14) \\ &= \int_{-\infty}^{2} \alpha x_1(\tau) d\tau + \int_{-\infty}^{2} \beta x_2(\tau) d\tau && (15) \\ &= \alpha y_1(t) + \beta y_2(t). && (16) \end{aligned}$$

Memorylessness: The system is NOT memoryless because the output at time t depends on input values at times other than t.

Time Invariance: The system is NOT time invariant. To see this, we let

$$\begin{aligned} x(t) &= u(t) - u(t-1) \text{ so that} && (17) \\ y(t) &= \int_{-\infty}^{2} x(\tau) d\tau && (18) \\ &= \int_{0}^{1} d\tau && (19) \\ &= 1, && (20) \end{aligned}$$

i.e. constant for all t, so in particular, $y(t-3) = 1$ for any value of t[2]. However, if we let

$$x_3(t) = x(t-3) = u(t-3) - u(t-4)$$

. Then

$$\begin{aligned} y_3(t) &= \int_{-\infty}^{2} x_3(\tau) d\tau && (21) \\ &= \int_{3}^{2} x(\tau) d\tau && (22) \\ &= 0 && (23) \\ &\neq y(t-3). && (24) \end{aligned}$$

[2] We need only a single delay value for which the property does not hold to show that the system is not time invariant, so 3 will do.

(c) $y(t) = (dx/dt)(t)$ ($\boxed{\text{causal}}$, invertible, $\boxed{\text{linear}}$, $\boxed{\text{memoryless}}$, $\boxed{\text{time invariant}}$)

Causality: The system is memoryless, hence causal.

Invertibility: The system is NOT invertible because any two inputs that differ by a constant yield the same output.

Linearity: The system is linear because if

$$y_1(t) = (dx_1/dt)(t), \text{ and} \tag{25}$$
$$y_2(t) = (dx_2/dt)(t), \tag{26}$$

and $x(t) = \alpha x_1(t) + \beta x_2(t)$, then the output $y(t)$ corresponding to the input $x(t)$ is

$$\begin{aligned} y(t) &= (d(\alpha x_1 + \beta x_2)/dt)(t) && (27)\\ &= \alpha(dx_1/dt)(t) + \beta(dx_2/dt)(t) && (28)\\ &= \alpha y_1(t) + \beta y_2(t). && (29) \end{aligned}$$

Memorylessness: The system is memoryless because the output at time t depends on input values at only time t.

Time Invariance: The system is time invariant. To see this, we let $y(t)$ be the output corresponding to the input $x(t)$ and let $x_a(t) = x(t-a)$. Then the output $y_a(t)$ corresponding to the input signal $x_a(t)$ is

$$y_a(t) = (dx_a/dt)(t) = (d(x)/dt)(t-a) = y(t-a). \tag{30}$$

(d) $y(t) = x(t/3)$ (causal, $\boxed{\text{invertible}}$, $\boxed{\text{linear}}$, memoryless, $\boxed{\text{time invariant}}$)

Causality: The system is NOT causal because if $t < 0$, then the output depends on future values of the input, e.g. for $t = 3$ we have $y(-3) = x(-1)$.

Invertibility: The system is invertible by applying the function $w(t) = y(3t)$.

Linearity: The system is linear because if

$$y_1(t) = x_1(t/3), \text{ and} \tag{31}$$
$$y_2(t) = x_2(t/3) \tag{32}$$

and $x(t) = \alpha x_1(t) + \beta x_2(t)$, then the output $y(t)$ corresponding to the input $x(t)$ is

$$\begin{aligned} y(t) &= (\alpha x_1 + \beta x_2)(t/3) && (33)\\ &= \alpha x_1(t/3) + \beta x_2(t/3) && (34)\\ &= \alpha y_1(t) + \beta y_2(t) && (35) \end{aligned}$$

Memorylessness: The system is NOT memoryless because the output at time t depends on input values at times other than t.

Time Invariance: The system is time invariant. To see this, we let $y(t)$ be the output corresponding to the input $x(t)$ and let $x_a(t) = x(t-a)$. Then the output $y_a(t)$ corresponding to the input signal $x_a(t)$ is

$$y_a(t) = x_a(t/3) = x((t-a)/3) = y(t-a) \tag{36}$$

(e) $y(t) = \cos(x(t))$ ($\boxed{\text{causal}}$, invertible, linear, $\boxed{\text{memoryless}}$, $\boxed{\text{time invariant}}$)

Causality: The system is memoryless, hence causal.

Invertibility: The system is NOT invertible, e.g. suppose that $x_1(t) = (\pi/2)u(t)$ and $x_2(t) = -(\pi/2)u(t)$. Then $y_1(t) = \cos(x_1(t)) = 0 = \cos(x_2(t)) = y_2(t), \forall t$.

Linearity: The system is NOT linear because if

$$\begin{aligned} y_1(t) &= \cos(x_1(t)), \text{ and} \qquad (37)\\ y_2(t) &= \cos(x_2(t)), \qquad (38)\end{aligned}$$

and $x(t) = \alpha x_1(t) + \beta x_2(t)$, then the output $y(t)$ corresponding to the input $x(t)$ is

$$\begin{aligned} y(t) &= \cos(\alpha x_1(t) + \beta x_2(t)) \qquad (39)\\ &= \cos(\alpha x_1(t))\cos(\beta x_2(t)) - \sin(\alpha x_1(t))\sin(\beta x_2(t)) \qquad (40)\\ &\neq \alpha\cos(x_1(t)) + \beta\cos(x_2(t)) \qquad (41)\end{aligned}$$

in general. (We used an identity from Chapter B of the texbook to get the second equality above.)

Memorylessness: The system is memoryless because the output at time t depends on input values at only time t.

Time Invariance: The system is time invariant. To see this, we let $y(t)$ be the output corresponding to the input $x(t)$ and let $x_a(t) = x(t-a)$. Then the output $y_a(t)$ corresponding to the input signal $x_a(t)$ is

$$y_a(t) = \cos(x_a(t)) = \cos(x(t-a)) = y(t-a). \tag{42}$$

2. For the system described by the differential equation

$$(D^2 + \alpha D + 6)y(t) = (D + 4)x(t) \tag{43}$$

(a) Suppose $\alpha = 5$.

i. Find the eigenvalues and modes.
Solution: Solving $Q(\lambda) = \lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0$, we obtain the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -3$. The corresponding modes are $c_1 e^{-2t}$ and $c_2 e^{-3t}$.

ii. Find the zero input response $y_0(t)$ corresponding to the initial conditions $y(0) = 1$, $\dot{y}(0) = 3$.
Solution: The ZIR is

$$y_0(t) = c_1 e^{-2t} + c_2 e^{-3t}, \text{ so} \tag{44}$$

$$\dot{y}_0(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}. \tag{45}$$

Solving

$$y_0(0) = 1 = c_1 + c_2 \text{ and} \tag{46}$$

$$\dot{y}_0(0) = 3 = -2c_1 - 3c_2 \tag{47}$$

for c_1 and c_2 yields

$$c_1 = 6 \text{ and } c_2 = -5$$

so

$$y_0(t) = 6e^{-2t} - 5e^{-3}t$$

iii. Find the impulse response $h(t)$.
Solution: First we solve for $y_n(t)$ by applying the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$ to the linear combination of the modes

$$y_n(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

Solving

$$y_n(0) = 0 = c_1 + c_2 \text{ and} \tag{48}$$

$$\dot{y}_n(0) = 1 = -2c_1 - 3c_2 \tag{49}$$

for c_1 and c_2 yields

$$c_1 = 1 \text{ and } c_2 = -1$$

so

$$y_n(t) = e^{-2t} - e^{-3t}, \text{ and} \tag{50}$$

$$\dot{y}_n(t) = -2e^{-2t} + 3e^{-3t}. \tag{51}$$

The impulse response will be given by

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]u(t). \tag{52}$$

The degree of $Q(D)$ being one greater than the degree of $P(D)$, the value of b_0 is zero. We obtain

$$\begin{aligned} h(t) &= [(D+4)y_n(t)]u(t) && (53)\\ &= [-2e^{-2t} + 3e^{-3t} + 4(e^{-2t} - e^{-3t})]u(t) && (54)\\ &= [2e^{-2t} - e^{-3t}]u(t). && (55) \end{aligned}$$

(b) Now suppose $\alpha = 1$.

i. Find the eigenvalues and modes.
Solution: Solving $Q(\lambda) = \lambda^2+\lambda+6 = (\lambda+\alpha+j\beta)(\lambda+\alpha-j\beta) = 0$, where $\alpha = -1/2$ and $\beta = \sqrt{23}/2$. we obtain the eigenvalues $\lambda_1 = -1/2+j\sqrt{23}/2$ and $\lambda_2 = -1/2 - j\sqrt{23}/2$. The corresponding modes are $c_1e^{\lambda_1 t}$ and $c_2e^{\lambda_2 t}$, or $ce^{\alpha t}\cos(\beta t+\theta)$, depending upon which representation one prefers.

ii. Find the zero input response $y_0(t)$ corresponding to the initial conditions $y(0) = 1$, $\dot{y}(0) = 3$.
Solution: For readability, I'll write the intermediate equations in terms of α and β and substitute for them at the end. The ZIR is

$$\begin{aligned} y_0(t) &= c_1e^{(\alpha+j\beta)t} + c_2e^{(\alpha-j\beta)t}, \text{ so} && (56)\\ \dot{y}_0(t) &= (\alpha+j\beta)c_1e^{(\alpha+j\beta)t} + (\alpha-j\beta)c_2e^{(\alpha-j\beta)t}. && (57) \end{aligned}$$

Solving

$$\begin{aligned} y_0(0) &= 1 = c_1 + c_2 \text{ and} && (58)\\ \dot{y}_0(0) &= 3 = (\alpha+j\beta)c_1 + (\alpha-j\beta)c_2 && (59) \end{aligned}$$

for c_1 and c_2 yields

$$c_1 = \frac{\beta - j(3-\alpha)}{2\beta} \text{ and } c_2 = \frac{\beta + j(3-\alpha)}{2\beta} \tag{60}$$

so

$$\begin{aligned} y_0(t) &= \frac{\beta - j(3-\alpha)}{2\beta}e^{(\alpha+j\beta)t} + \frac{\beta + j(3-\alpha)}{2\beta}e^{(\alpha-j\beta)t} && (61)\\ &= e^{\alpha t}\left(\frac{e^{j\beta t}+e^{-j\beta t}}{2}\right) + \frac{j(3-\alpha)e^{\alpha t}}{\beta}\left(\frac{-e^{j\beta t}+e^{-j\beta t}}{2}\right) && (62)\\ &= e^{\alpha t}\cos(\beta t) + \frac{(3-\alpha)e^{\alpha t}}{\beta}\sin(\beta t) && (63)\\ &= e^{-t/2}\cos(\sqrt{23}t/2) + \frac{7e^{-t/2}}{\sqrt{23}}\sin(\sqrt{23}t) && (64) \end{aligned}$$

iii. Find the impulse response $h(t)$.

Solution: First we solve for $y_n(t)$ by applying the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$ to the linear combination of the modes

$$y_n(t) == c_1 e^{(\alpha+j\beta)t} + c_2 e^{(\alpha-j\beta)t}$$

Solving

$$\begin{aligned} y_n(0) &= 0 = c_1 + c_2 \text{ and} && (65) \\ \dot{y}_n(0) &= 1 = (\alpha + j\beta)c_1 + (\alpha - j\beta)c_2 && (66) \end{aligned}$$

for c_1 and c_2 yields

$$c_1 = -j\frac{1}{2\beta} \text{ and } c_2 = j\frac{1}{2\beta}$$

so

$$\begin{aligned} y_n(t) &= j\frac{1}{2\beta}(-e^{(\alpha+j\beta)t} + e^{(\alpha-j\beta)t}) && (67) \\ &= e^{\alpha t}\left(\frac{\sin(\beta t)}{\beta}\right), \text{ and} && (68) \\ \dot{y}_n(t) &= \alpha e^{\alpha t}\left(\frac{\sin(\beta t)}{\beta}\right) + e^{\alpha t}\cos(\beta t) && (69) \end{aligned}$$

The impulse response is then

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]u(t). \quad (70)$$

The degree of $Q(D)$ being one greater than the degree of $P(D)$, the value of b_0 is zero. We obtain

$$\begin{aligned} h(t) &= [(D+4)y_n(t)]u(t) && (71) \\ &= \left[(\alpha+4)e^{\alpha t}\left(\frac{\sin(\beta t)}{\beta}\right) + e^{\alpha t}\cos(\beta t)\right]u(t) && (72) \\ &= 7e^{-t/2}\left(\frac{\sin(\sqrt{23}t)}{\sqrt{23}}\right) + e^{-t/2}\cos(\sqrt{23}t). && (73) \end{aligned}$$

(c) Finally, suppose $\alpha = 2\sqrt{6}$.

i. Find the eigenvalues and modes.
Solution: Solving $Q(\lambda) = \lambda^2 + 2\sqrt{6}\lambda + 6 = (\lambda + \sqrt{6})^2 = 0$, we obtain the eigenvalues $\lambda_1 = \lambda_2 = -\sqrt{6}$. The corresponding modes are $c_1 e^{-\sqrt{6}t}$ and $c_2 t e^{-\sqrt{6}t}$.

ii. Find the zero input response $y_0(t)$ corresponding to the initial conditions $y(0) = 1$, $\dot{y}(0) = 3$.

Solution: The ZIR is

$$
\begin{aligned}
y_0(t) &= c_1 e^{-\sqrt{6}t} + c_2 t e^{-\sqrt{6}t}, \text{ so} & (74)\\
\dot{y}_0(t) &= -\sqrt{6}(c_1 e^{-\sqrt{6}t} + c_2 t e^{-\sqrt{6}t}) + c_2 e^{-\sqrt{6}t}. & (75)
\end{aligned}
$$

Solving

$$
\begin{aligned}
y_0(0) &= 1 = c_1 \text{ and} & (76)\\
\dot{y}_0(0) &= 3 = -\sqrt{6}c_1 + c_2 & (77)
\end{aligned}
$$

for c_1 and c_2 yields

$$c_1 = 1 \text{ and } c_2 = 3 + \sqrt{6}$$

so

$$y_0(t) = (1 + (3 + \sqrt{6})t)e^{-\sqrt{6}t}.$$

iii. Find the impulse response $h(t)$.

Solution: First we solve for $y_n(t)$ by applying the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$ to the linear combination of the modes

$$y_n(t) = (c_1 + c_2 t)e^{-\sqrt{6}t}.$$

Solving

$$
\begin{aligned}
y_n(0) &= 0 = c_1 \text{ and} & (78)\\
\dot{y}_n(0) &= 1 = -\sqrt{6}c_1 + c_2 & (79)
\end{aligned}
$$

for c_1 and c_2 yields

$$c_1 = 0 \text{ and } c_2 = 1$$

so

$$
\begin{aligned}
y_n(t) &= t e^{-\sqrt{6}t}, \text{ and} & (80)\\
\dot{y}_n(t) &= -\sqrt{6} t e^{-\sqrt{6}t} + e^{-\sqrt{6}t} & (81)
\end{aligned}
$$

The impulse response is then

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]u(t). \quad (82)$$

The degree of $Q(D)$ being one greater than the degree of $P(D)$, the value of b_0 is zero. We obtain

$$
\begin{aligned}
h(t) &= [(D + 4)y_n(t)]u(t) & (83)\\
&= [(-\sqrt{6}t + 1)e^{-\sqrt{6}t} + 4te^{-\sqrt{6}t}]u(t) & (84)\\
&= [(1 + (-\sqrt{6} + 4)t)e^{-\sqrt{6}t}]u(t). & (85)
\end{aligned}
$$