ECE 602 Lecture Notes: 
Examination of a Companion Matrix

Yesterday in lecture we examined the companion matrix

\[ A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \]

(1)

The characteristic equation of \( A \) is

\[
\det(sI - A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} - \frac{1}{\alpha_4} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{vmatrix}.
\]

(3)

Thus, if \( \alpha_4 \) were zero, the characteristic polynomial would be

\[ s \left( s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 \right) \]

and zero would be an eigenvalue of \( A \). If zero were an eigenvalue of \( A \), then \( A \) would be singular, i.e. non-invertible.

Can the matrix be singular if \( \alpha_4 \) is nonzero? The answer is “no”, because if \( \alpha_4 \) is nonzero, then the four rows of the matrix \( A \) are linearly independent regardless of the values of \( \alpha_1, \alpha_2, \) and \( \alpha_3 \). Of course we could also have used this simpler argument to establish that if \( \alpha_4 \) were zero, the matrix \( A \) would be singular. Specifically, if \( \alpha_4 \) were zero, the fourth column would be the zero vector so we would have only three linearly independent rows/columns.

To show that

\[ A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix}, \]

(3)

we simply multiply \( A \) by \( A^{-1} \) and verify that the result is \( I \).

What can we say about the eigenvalues and eigenvectors?
Suppose that $\lambda$ is an eigenvalue of $A$. Then

$$\lambda^4 + \alpha_1 \lambda^3 + \alpha_2 \lambda^2 + \alpha_3 \lambda + \alpha_4 = 0$$

Also,

$$0 = (A - \lambda I) v$$

$$= \begin{bmatrix}
-\alpha_1 - \lambda & -\alpha_2 & -\alpha_3 & -\alpha_4 \\
1 & -\lambda & 0 & 0 \\
0 & 1 & -\lambda & 0 \\
0 & 0 & 1 & -\lambda
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix}$$

$$= \begin{bmatrix}
(-\alpha_1 - \lambda)v_1 - \alpha_2 v_2 - \alpha_3 v_3 - \alpha_4 v_4 \\
v_1 - \lambda v_2 \\
v_2 - \lambda v_3 \\
v_3 - \lambda v_4
\end{bmatrix}$$

(4)

Here we have (assuming $\lambda$ known) four equations in the four unknowns $v_1, v_2, v_3, \text{ and } v_4$. From the fourth row we have that $v_3 = \lambda v_4$. From the third, $v_2 = \lambda v_3 = \lambda^2 v_4$. From the second, $v_1 = \lambda v_2 = \lambda^2 v_3 = \lambda^3 v_4$. Then the first row is

$$0 = (-\alpha_1 - \lambda)v_1 - \alpha_2 v_2 - \alpha_3 v_3 - \alpha_4 v_4$$

$$= (-\alpha_1 - \lambda) \lambda^3 v_4 - \alpha_2 \lambda^2 v_4 - \alpha_3 \lambda v_4 - \alpha_4 v_4$$

$$= ((-\alpha_1 - \lambda) \lambda^3 - \alpha_2 \lambda^2 - \alpha_3 \lambda - \alpha_4) v_4$$

$$= - (\lambda^4 + \alpha_1 \lambda^3 + \alpha_2 \lambda^2 + \alpha_3 \lambda + \alpha_4) v_4.$$  

(5)

We already know that if $\lambda$ is an eigenvalue of $A$ it satisfies the characteristic equation of $A$ so this first equation holds regardless of the value of $v_4$. As always, we can choose one element of the eigenvector arbitrarily so let’s make $v_4 = 1$. Then the eigenvector corresponding to $\lambda$ is

$$v = [\lambda^3 \lambda^2 \lambda 1]^T.$$