Discussion of Problem A-6-4

For a system to be completely state controllable and completely observable, its pulse transfer function must not have any pole-zero cancellations. This is shown for the limited case of SISO systems with constant bias in the output described by

\[ x(k + 1) = Gx(k) + Hu(k) \]
\[ y(k) = Cx(k) + D. \]

We know that the pulse transfer function for this system is

\[ F(z) = C(zI - G)^{-1}H + D. \]

**Claim:** If the above system is completely state controllable and completely observable then there is no pole-zero cancellation in the pulse transfer function \( F(z) \).

**Proof:** We suppose that the above system is completely state controllable and completely observable and that there is pole-zero cancellation in the pulse transfer function \( F(z) \). Showing that this leads to a contradiction will prove the claim.

In order to obtain an expression for the pulse transfer function, we will take the determinant of both sides of the identity

\[
\begin{bmatrix}
    I & 0 \\
    C(zI - G)^{-1} & 1
\end{bmatrix}
\begin{bmatrix}
zI - G & H \\
-C & D
\end{bmatrix}
= \begin{bmatrix}
zI - G & H \\
0 & F(z)
\end{bmatrix}.
\]

The determinant of a product of square matrices is the product of the determinants of the individual matrices and the first matrix on the left hand side has determinant one, so we have

\[
\begin{vmatrix}
zI - G & H \\
-C & D
\end{vmatrix}
= |zI - G|F(z),
\]

which can be solved for the pulse transfer function

\[
F(z) = \frac{|zI - G|}{|zI - Q|}.
\]

Now we suppose that \( z = z_1 \) is a root of both the numerator and the denominator of the pulse transfer function. Examining the denominator polynomial, we see that it is the characteristic polynomial of the matrix \( G \), so \( z_1 \) is an eigenvalue of \( G \) and has a corresponding eigenvector, which we’ll call \( v_1 \). Next we consider the numerator polynomial. If we define the matrix

\[
Q = \begin{bmatrix}
    G & -H \\
    C & z_1 - D
\end{bmatrix}
\]

then

\[
|z_1I - Q| = \begin{vmatrix}
z_1I - G & H \\
-C & D
\end{vmatrix}.
\]
so $Q$ has a (nonzero) eigenvector $q_1$ corresponding to eigenvalue $z_1$. Now we can construct the vector $[v^T \ w^T]^T$ given on page 480. The column vector $v$ consists of the first $n$ elements of $q_1$ and $w$ is the last element, so

$$0 = [z_1I - Q]q_1 = \begin{bmatrix} z_1I - G & H \\ -C & D \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ (9)

The proof now proceeds as indicated in the text. We consider two cases: $w = 0$ and $w \neq 0$.

**Case $w \neq 0$** From (9),

$$(G - z_1I)v = Hw.$$ (10)

$z_1$ being a root of the characteristic equation of $G$, the characteristic polynomial $\phi_G(z)$ can be factored as

$$0 = \phi_G(z) = \hat{\phi}_G(z)(z - z_1)$$ (11)

and by the Cayley-Hamilton theorem we also have that

$$0 = \phi_G(G) = \hat{\phi}_G(G)(G - z_1I).$$ (12)

Thus, post-multiplying by $v$ and using (10) we have

$$0 = \phi_G(G)v = \hat{\phi}_G(G)(G - z_1I)v = \hat{\phi}_G(G)Hw.$$ (13)

Since we assumed $w \neq 0$, we must have $\hat{\phi}_G(G)H = 0$. Now since $\phi_G(G)$ is a matrix polynomial of degree $n$ in $G$, $\hat{\phi}_G(G)$ is a matrix polynomial of degree $n - 1$ in $G$, so the matrix polynomial $\hat{\phi}_G(G)H$ is a linear combination of the vectors $G^iH$ where $i \in \{1, 2, \ldots, n - 1\}$. This linear combination being equal to zero, implies that the matrix $M = [H \ G H \ \cdots \ G^{n-1}H]$ does not have full rank and thus the system is not completely state controllable, contradicting our assumption.

**Case $w = 0$** In this case, since $q_1 \neq 0$, we must have $v \neq 0$. (In fact, $v = v_1$, the eigenvector of $G$ corresponding to eigenvalue $z_1$ as defined earlier.) Then (9) simplifies to

$$(z_1I - G)v = 0$$ (14)

$$Cv = 0.$$ (15)

Taking the conjugate transpose we obtain

$$v^*G^* = z_1v^*$$ (16)

$$v^*C^* = 0$$ (17)

hence

$$v^*(G^*)^iC^* = v^*G^*(G^*)^{i-1}C^* = z_1v^*(G^*)^{i-1}C^* = \cdots = z_1^i v^*C^* = 0$$ (18)

and thus

$$v^*[C^* \ G^*C^* \ \cdots \ (G^*)^{n-1}C^*] = 0$$ (19)

so the rank of the observability matrix is less than $n$ and the system is not completely observable, contradicting our assumption.