Discretization of Continuous Time State Space Systems

Suppose we are given the continuous time state space system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  
\[ y(t) = Cx(t) + Du(t) \]

and apply an input that changes only at discrete (equal) sampling intervals. It would be nice if we could find matrices \( G \) and \( H \), independent of \( t \) or \( k \) so that we could obtain a discrete time model of the system,

\[ x((k+1)T) = G(T)x(kT) + H(T)u(kT) \]  
\[ y(kT) = Cx(kT) + Du(kT). \]

We will now determine the values of the matrices \( G \) and \( H \). It will turn out that while they are constant for a particular sampling interval, they depend on the value of the sampling interval, so for that reason I have written them as \( G(T) \) and \( H(T) \) in (3) above.

We start by using the solution of (1) to calculate the values of the state \( x \) at times \( kT \) and \( (k+1)T \). These are

\[ x((k+1)T) = e^{A(k+1)T}x(0) + e^{A(k+1)T}\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau \]  
\[ x(kT) = e^{AkT}x(0) + e^{AkT}\int_0^{kT} e^{-A\tau}Bu(\tau)d\tau. \]

We want to write \( x((k+1)T) \) in terms of \( x(kT) \) so we multiply all terms of (6) by \( e^{AT} \) and solve for \( e^{A(k+1)T}x(0) \), obtaining

\[ e^{A(k+1)T}x(0) = e^{AT}x(kT) - e^{A(k+1)T}\int_0^{kT} e^{-A\tau}Bu(\tau)d\tau. \]

Substituting for \( e^{A(k+1)T}x(0) \) in (5) we obtain

\[ x((k+1)T) = e^{AT}x(kT) + e^{A(k+1)T}\left[\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau - \int_0^{kT} e^{-A\tau}Bu(\tau)d\tau\right] \]

which, by linearity of integration, is equivalent to

\[ x((k+1)T) = e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau. \]

Next, we notice that within the interval from \( kT \) to \( (k+1)T \), \( u(t) = u(kT) \) is constant, as is the matrix \( B \), so we can take them out of the integral to obtain

\[ x((k+1)T) = e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}Bu(kT)d\tau \quad \tau \in [kT, (k+1)T). \]

We can take the \( e^{A(k+1)T} \) inside the integral to obtain

\[ x((k+1)T) = e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A[k(k+1)T-\tau]}Bu(kT) \quad \tau \in [kT, (k+1)T). \]
Now we see that as $\tau$ ranges from $kT$ to $(k+1)T$ (the lower to the upper limit of integration) the exponent of $e$ ranges from $T$ to 0. Accordingly, let’s define a new variable $\lambda = (k+1)T - \tau$. Then $d\lambda = -d\tau$ and $\lambda$ ranges from $T$ to 0 as $\tau$ ranges from $kT$ to $(k+1)T$. Thus we have

$$
x((k+1)T) = e^{AT}x(kT) - \int_0^T e^{A\lambda}d\lambda Bu(kT) \quad \lambda \in [0,kT),
$$

or

$$
x((k+1)T) = e^{AT}x(kT) + \int_0^T e^{A\lambda}d\lambda Bu(kT) \quad \lambda \in [0,kT).
$$

We see that in (13) we have written the state update equation exactly in the form of (3) where

$$
G(T) = e^{AT}
$$

(14)

$$
H(T) = \int_0^T e^{A\lambda}d\lambda B,
$$

(15)

so we’re done . . . except that we’d rather not leave the expression for $H(T)$ in the form of an integral. So long as $A$ is invertible, we can easily integrate, using the fact that

$$
\frac{d}{dt}e^{AT} = Ae^{AT} = e^{AT}A
$$

(16)

to obtain

$$
H(T) = A^{-1} \int_0^T A e^{A\lambda}d\lambda B = A^{-1}e^{AT}|_{\lambda=0}B
$$

(17)

$$
= A^{-1}(e^{AT} - I)B = (e^{AT} - I)BA^{-1}.
$$

(18)

Finally, note that while I restricted the value of $\tau$ and $\lambda$ to lie within a single sampling interval, $k$ appears nowhere in the expressions for $G(T)$ and $H(T)$. Our solution to (3) is thus

$$
x(kT) = (G(T))^k x(0) + \sum_{j=0}^{k-1} (G(T))^{k-j-1}H(T)u(jT), \quad k = 1, 2, 3, \ldots
$$

(19)

and we can see that at the sampling instants $kT$, this has exactly the same value as is obtained using (1). Specifically,

$$
(G(T))^k = (e^{AT})^k = e^{AkT},
$$

(20)

and since the input $u(t)$ is constant on sampling intervals,

$$
e^{AkT} \int_0^{kT} e^{-A\tau} Bu(\tau)d\tau = \sum_{j=0}^{k-1} e^{A(k-j-1)T} A^{-1}(e^{AT} - I)Bu(jT)
$$

(21)

$$
= \sum_{j=0}^{k-1} (G(T))^{k-j-1}H(T)u(jT).
$$

(22)